

# ON COLIMITS IN VARIOUS CATEGORIES OF MANIFOLDS

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## 1. INTRODUCTION

Homotopy theorists have traditionally studied topological spaces and used methods which rely heavily on computation, construction, and induction. Proofs often go by constructing some horrendously complicated object (usually via a tower of increasingly complicated objects) and then proving inductively that we can understand what's going on at each step and that in the limit these steps do what is required. For this reason, a homotopy theorist need the freedom to make any construction he wishes, e.g. products, subobjects, quotients, gluing towers of spaces together, dividing out by group actions, moving to fixed points of group actions, etc.

Over the past 50 years tools have been developed which allow the study of many different categories. The key tool for this is the notion of a model category, which is the most general place one can do homotopy theory. Model categories were introduced by the Fields Medalist Dan Quillen, and the best known reference for model categories is the book *Model Categories* by Mark Hovey. Model categories are defined by a list of axioms which are all analogous to properties that the category *Top* of topological spaces satisfies. The most important property of a model category is that it has three distinguished classes of morphisms—called weak equivalences, cofibrations, and fibrations—which follow rules similar to the classes of maps with those names in the category *Top* (so the homotopy theory comes into play by inverting the weak equivalences to make them isomorphisms). The fact that homotopy theorists need to be able to do lots of constructions is contained in the axiom that  $\mathcal{M}$  must be complete and cocomplete. We'll discuss these terms in more detail later on.

Based on the fact that all the model category axioms come from topological spaces, one might think that the only examples of model categories are topological things like simplicial sets. It's true that these categories are examples of model categories, but the surprising fact is that lots of other categories with no a priori connection to topology also admit model category structures. For example, the category chain complexes of modules over a ring admits a model structure, and this has been used to do computations which were not possible without it (including the notion of Andre-Quillen cohomology). Another example is the category of sets (which admits exactly 9 model structures), the topos of directed graphs, most any presheaf category, the theory of flows in computer science, and a plethora of abelian categories which admit *Ext* and *Tor* functors which behave nicely (c.f. Hovey's theory of Cotorsion Pairs). The purpose of this paper is to explore how one might do homotopy theory on manifold and Lie groups, and in particular to explore the notion of cocompleteness in these categories. We'll begin with the requisite category theory, then discuss the work of a different Fields Medalist on finding a model structure on the category of schemes, and then discuss how one might do something analogous for manifolds and Lie groups (and some alternative ways to get around the cocompleteness issue).

## 2. CATEGORY THEORY BACKGROUND

Let  $C$  be a category (i.e. it has a class of objects and a class of morphisms satisfying the usual rules). Let  $J$  be a small category (i.e.  $\text{ob}(J)$  and  $\text{mor}(J)$  are sets rather than classes). A **diagram** of type  $J$  in  $C$  is a

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*Date:* December 17, 2012.

functor  $F : J \rightarrow C$ . We say  $J$  is the indexing category for the diagram. This definition means  $F$  picks out an object of  $C$  for each  $j \in \text{ob}(J)$  and then assigns  $C$ -morphisms  $F(f)$  for each  $J$ -morphism. For example, here is an example for  $J$  and the corresponding diagram of type  $J$  in some  $C$ :

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ \downarrow g & & \\ \bullet & & \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{F(f)} & B \\ \downarrow F(g) & & \\ C & & \end{array}$$

Here  $A, B, C \in C$  are the images of the corresponding  $J$ -objects on the left. Here are two more examples:

$$\begin{array}{ccc} \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots & & A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \dots \\ & & \bullet \xrightarrow{\quad} \bullet \qquad \qquad A \xrightarrow{\quad} B \end{array}$$

Returning to the general picture, let  $F$  be a diagram of type  $J$  in  $C$ . Informally, the colimit of such a diagram is the object of  $C$  which the diagram is “trying to be” in the same way that a limit of a sequence is the number that the sequence is trying to be. Formally, the **colimit** of  $F$  is an object  $L \in C$  which has maps  $\phi_X : F(X) \rightarrow L$  for each  $X \in J$  such that for any  $f : X \rightarrow Y$  in  $J$ ,  $\phi_Y \circ F(f) = \phi_X$ . Furthermore, this object  $L$  is **universal** in the sense that if  $N$  is any other object in  $C$  which satisfies the condition we just placed on  $L$  (i.e. has maps from all elements in the diagram, commuting with the diagram’s structure maps) then there is a unique map  $L \rightarrow N$ . This universality is saying that  $L$  is the “closest” object the diagram maps to. In the first example above, the diagram with colimit included and universality demonstrated looks like:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow h & & \downarrow g \\ C & \xrightarrow{k} & B \coprod_A C \\ & \searrow & \downarrow \\ & & N \end{array}$$

(Note: A curved arrow from  $A$  to  $N$  and a dotted arrow from  $B \coprod_A C$  to  $N$  are also present in the original diagram.)

Here the dotted arrow exists by universality because of the existence of the curved arrows. This diagram is commutative, which means the two paths from  $A$  to  $B \coprod_A C$  are equal (i.e.  $g \circ f = k \circ h$ ), the two from  $B$  to  $N$  are equal, and the two from  $C$  to  $N$  are equal. The object  $B \coprod_A C$  is called the **pushout** of  $f$  and  $h$ , and the arrow  $\searrow$  is notation to let us know this is a pushout diagram. Analyzing pushout diagrams is part of the bread-and-butter work of anyone studying model categories and operads. The object  $B \coprod_A C$  is a copy of  $B$  glued together with a copy of  $C$  over the copy of  $A$  which sits in both. Thinking of pushouts and colimits as gluing can be a useful point of view, and helps clarify why homotopy theorists want categories with all colimits. The symbol  $\coprod$  without a subscript means disjoint union, and is known as the **coproduct**.

The colimit of the second example above is called the **direct limit** or inductive limit by algebraists. This is a confusing piece of terminology because it means the inverse limit in algebra is then an example of a categorical **limit** (where all arrows are reversed so it’s the “closest” object which maps to the diagram). The easiest way for me to remember the difference between colimits and limits is to remember that coproduct is a colimit and product is a limit. The product maps to both factors, just like the limit does. The coproduct is usually the disjoint union of the objects, and all objects map to it by inclusion. A category is called **cocomplete** if it has colimits for all diagrams coming from small indexing categories  $J$ .

The colimit in the third example is called the **coequalizer** of the two arrows. The diagram with the colimit included looks like  $A \rightrightarrows B \rightarrow C$ . One place this type of colimit arises in practice is the tensor product. If  $R$  is a commutative ring and  $M$  and  $N$  are  $R$ -modules, then we can compute the tensor product  $M \otimes_R N$  as the

coequalizer of the diagram  $M \amalg R \amalg N \rightrightarrows M \amalg N$  where one of the two maps is the right-action of  $R$  on  $M$  (i.e. takes  $M \amalg R \rightarrow M$ ) and the other is the left-action of  $R$  on  $N$  (taking  $R \amalg N \rightarrow N$ ). The universal property of coequalizer exactly says that one can move multiplication by  $r \in R$  across the tensor product, i.e.  $m \otimes rn = mr \otimes n$ .

A final example of colimit which we'll discuss in more detail describes a group action on an object of  $C$ . So let  $G$  be a finite group and let the indexing category  $J$  be the category  $G$  which has one object,  $|G|$  arrows (one for each element of the group), and composition law given by the group law (i.e.  $f_\sigma \circ f_\tau = f_{\sigma\tau}$ ). In this case a functor  $F : G \rightarrow C$  simply picks out one object  $M \in C$  and lets  $G$  act on  $M$  via which automorphisms of  $M$  are assigned to which group elements  $g \in G$ . The definition of functor assures us that the action law  $g_1 \cdot (g_2 \cdot m) = (g_1 * g_2) \cdot m$  is satisfied. In the category of smooth manifolds, the fact that  $G$  sits inside  $\text{Aut}(M)$  tells us  $G$  is isomorphic to a group of diffeomorphisms, so the action map is necessarily smooth even though the group  $G$  need not be a Lie group or have a smooth structure. The colimit of this diagram is the object  $M / \sim$  where we have identified  $m_1$  and  $m_2$  if there is some  $g$  such that  $g \cdot m_1 = m_2$ . This object is also known as the **orbit space** or **space of coinvariants**  $M_G$ , and is an important object in equivariant homotopy theory.

### 3. COLIMITS AND SCHEMES

In 2002 Vladimir Voevodsky won the Field Medal for his proof of the Milnor Conjecture ([9]). The Milnor Conjecture relates the Milnor K-theory (mod 2) of a field  $F$  of characteristic  $\neq 2$  to the Galois cohomology of  $F$  with coefficients in  $\mathbb{Z}/2\mathbb{Z}$ . Voevodsky's proof required him to be able to do homotopy theory in the category of schemes, so he could do motivic cohomology on abelian varieties. One problem is that this category is not cocomplete. For example, [4] shows that there is colimit to the diagram  $\text{Spec}(k(t)) \rightrightarrows \mathbb{A}^1 \amalg \mathbb{A}^1$  in the category of schemes because if it did exist the colimit would be  $\mathbb{A}^1$  with every closed point doubled, which is too non-separated to be a scheme. Interestingly, the category of affine schemes over a field  $k$  does have all small colimits because its opposite category  $k$ -algebras has all limits. However, even in this case one does not get the "right" colimits because the colimits miss important geometric information (see [3], Example 2.1.1).

In order to fix this problem, Voevodsky had to embed the category  $Sm/k$  of smooth schemes over  $k$  into the category of presheaves of sets on  $Sm/k$ , via the Yoneda embedding. This category is complete and cocomplete. Applying the sheafification functor with respect to the Nisnevich topology on  $Sm/k$  gives another complete and cocomplete category  $Spc$ . We can see  $X \in Sm/k$  sitting inside  $Spc$  as the colimit of any Nisnevich cover of  $X$ . The maps  $X \times \mathbb{A}^1 \rightarrow X$  generate the  $\mathbb{A}^1$ -weak equivalences of  $Spc$ , where cofibrations are monomorphisms. The Nisnevich topology is a special case of the notion of a Grothendieck topology on a category. It sits between the Zariski and tale topologies, which also yield model structures. Note that this is the treatment given by [11], whereas originally Voevodsky ([10]) used the category of simplicial sheaves of sets in the Nisnevich topology on  $Sm/k$  and defined weak equivalences by way of the functor of points approach to schemes. He then gets the  $\mathbb{A}^1$ -weak equivalences above by applying Bousfield Localization.

### 4. COLIMITS, SMOOTH MANIFOLDS, AND LIE GROUPS

Let  $Mnfd$  be the category of smooth manifolds,  $Top$  the category of topological spaces, and  $LieGp$  the category of Lie groups. In order to do homotopy theory with manifolds, we'd like to know if  $Mnfd$  is closed under colimits. The answer is no, and this should not come as a surprise since smooth schemes are the algebraic analogue of smooth manifolds and are so poorly behaved. An explicit example is given in [7]:

**Example 1.** *The pushout of  $\mathbb{R} \leftarrow \{0\} \rightarrow \mathbb{R}$  is the union of the coordinate axes in  $\mathbb{R}^2$  (i.e. it looks like a cross). To show this, let  $P$  denote the union of the coordinate axes. Certainly the diagram maps to  $P$ , since we can include the left copy of  $\mathbb{R}$  as the  $x$ -axis (call this map  $i$ ), the right copy as the  $y$ -axis (call this map  $j$ ), and  $\{0\}$  as  $\{(0, 0)\}$  where the two copies agree. If the diagram maps to any other space  $Y$  (say by maps  $f$  from the left copy of  $\mathbb{R}$  and  $g$  from the right copy) then there is a unique map from  $P$  to  $Y$  by sending a point  $(i(a), 0)$  to  $f(a)$  and sending  $(0, j(b))$  to  $g(b)$ . This proves that  $P$  has the universal property of the pushout, so  $P$  is the pushout. However,  $P$  is clearly not a manifold because the origin doesn't have a locally euclidean neighborhood.*

There is one subtlety which the proof above leaves out. We computed the pushout *in the category of topological spaces* and saw that it's not a manifold. But how do we know that the pushout in the category of manifolds is the same as the pushout in the category of topological spaces? This is addressed in [7] and is not very difficult. One must simply show that the map from the (manifold) pushout  $X$  into  $\mathbb{R}^2$  which takes  $X$  to  $P$  is an initial map in the category of Froelicher spaces (which sits between *Mnfd* and *Top*). The details only take one paragraph, but would take us away from our goal of understanding pushouts in *Mnfd* so we'll omit them. The point of this example is that none of the following categories are cocomplete: (topological) manifolds, smooth manifolds, complex analytic spaces, symplectic manifolds, and Lie Algebras. This is unfortunate, because we often want to take quotients by submanifolds, glue manifolds together, or divide out by group actions.

One way to fix this is to proceed as Voevodsky did and embed the category of manifolds as a full subcategory of a cocomplete category. For *Mnfd* to be a **full subcategory** of some  $C$  simply means that all the morphisms between two manifolds  $M$  and  $N$  are still morphisms between their images in  $C$ . This seems like too much work, so perhaps a better way to fix this problem is to try and figure out which colimits the category *Mnfd* does have. Perhaps if the manifolds and diagrams are sufficiently nice then colimits will again be manifolds (e.g. if a group acts nicely then perhaps the orbit space will be a manifold). Still a third way (if we only care about colimits which come from group actions) is to change what we mean by a group action. This leads to the notion of an orbifold. We consider these three approaches in more detail below.

**4.1. Specific Colimits of Interest in *Mnfd*.** Before jumping into a Voevodsky-style embedding, let's focus on specific diagrams and especially nice manifolds to see what types of colimits we can get without leaving the category *Mnfd*. In order to do this, we might need to restrict to a subclass of nicely behaved manifolds and a subcollection of nice diagrams. One place this kind of philosophy came up historically was the introduction of homogeneous spaces as manifolds  $M$  which have a transitive action of a group  $G$ . However, what one does with homogeneous manifolds is not to mod out by the group action (which would yield  $M/G$ ) but rather to realize  $M$  as  $G/K$  where  $K$  is the stabilizer of a point. This is also a colimit construction, and tells us that homogeneous manifolds are the manifolds which can be realized as a (nice) colimit in the category of Lie Groups (since  $K$  is a Lie subgroup of  $G$  which acts on  $G$  by left multiplication).

A better example of the type of result we're seeking is the small point that the coproduct (disjoint union) of two manifolds of dimension  $d$  is a manifold of dimension  $d$ , but the coproduct of manifolds of different dimension is not a manifold. In the pushout example above we also had manifolds of different dimensions appearing (namely  $\{0\}$  is 0-dimensional while  $\mathbb{R}$  is 1-dimensional). This leads to the conjecture that diagrams with manifolds of the same dimension might have colimits which are manifolds. This turns out to be false, as the following example (from [8]) shows:

**Example 2.** *Let  $\mathbb{R}^*$  denote  $\mathbb{R} - \{0\}$ . Consider the diagram  $\mathbb{R}^* \amalg \mathbb{R}^* \rightrightarrows \mathbb{R} \amalg \mathbb{R}$  where one of the two arrows sends the first  $\mathbb{R}^*$  to the first  $\mathbb{R}$  and the second  $\mathbb{R}^*$  to the second, and the other arrow swaps the two. The colimit of this diagram is the coequalizer of these two arrows, i.e. the smallest thing  $\mathbb{R} \amalg \mathbb{R}$  can map into which makes the two arrows equal each other. This object is two copies of  $\mathbb{R}$  glued together along the subset*

$\mathbb{R}^*$ . Topologists know this space as the “line with two origins” because that’s what it looks like. This is known to be a non-Hausdorff space, so can’t be a manifold.

As with the previous example, to be fully rigorous, one would have to show that the pushout in  $Top$  matches that in  $Mnfd$ , by way of Froelicher spaces. We’ll omit this detail now and forever, because the method is the same as in [7] and the details are there. This example shows that even very nice diagrams (e.g. with only two objects and morphisms which are open injections) of manifolds of the same dimension can fail to have colimits. The only way this example could have been nicer is if the manifolds were connected (since  $\mathbb{R}^*$  is not). However, the same logic works for the diagram  $\mathbb{C}^* \rightrightarrows \mathbb{C}$  where one map is inclusion and the other is inclusion followed by  $z \mapsto 1/z$ . Again, the result is not Hausdorff because the images of  $z = 1$  and  $z = -1$  cannot be separated (any ball containing  $z = 1$  will intersect any ball containing  $z = -1$  after the gluing).

So far we’ve shown that  $Mnfd$  is poorly behaved with respect to even very nice pushouts and coequalizers. A simple example to show it’s poorly behaved with respect to direct limits is the sequence  $\mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3 \rightarrow \dots$ , where the colimit  $\mathbb{R}^\infty$  is not a manifold. The only colimits of interest left are the ones which come from group actions. The subject of group actions on manifolds has been studied much more than the other types of colimits. For example, in Riemannian Geometry these considerations give a slick proof that  $\pi_1(M)$  is countable for any manifold  $M$ , by viewing elements of  $\pi_1$  as acting via Deck Transformations. If we plan to mod out by the group action, then for point-set topological reasons (i.e. to get the quotient to be Hausdorff) we need to know that the action is **properly discontinuous**. This means the action is proper, i.e.  $G \times M \rightarrow M \times M$  via  $(g, m) \mapsto (m, gm)$  is a proper map (inverse images of compact sets are compact) and that  $G$  is discrete.

Let us first consider group actions where the group  $G$  is not a Lie group (though of course we must still have  $G \leq Aut(M) = Diffeo(M, M)$ ). We’ll assume the action is proper, so that  $M/G$  is Hausdorff. Quotients of second countable spaces need not be second countable, but open quotients (i.e. when  $\pi : M \rightarrow M/G$  is open) are. Because of how  $M/G$  is defined, the quotient map is always open. To see this, note that  $\pi^{-1}(\pi(U)) = \bigcup_g gU$  is open and  $\pi$  is a quotient map, so  $\pi(U)$  is open in  $M/G$ . However, even though  $M/G$  is Hausdorff and second countable, there are many proper actions where  $M/G$  is not a manifold (not even a topological manifold). An example can be found in [2]:

**Example 3.** Let  $\mathbb{Z}/2\mathbb{Z}$  act on  $\mathbb{R}P^3$  by the antipodal map. Then  $\mathbb{R}P^3/G \cong C(\mathbb{R}P^2)$ , the cone on  $\mathbb{R}P^2$ . To see this, let points on  $C(\mathbb{R}P^2)$  be denoted by  $([x, y, z], t)$  where  $\mathbb{R}P^2 \times \{0\}$  is the cone point  $p$ . The map  $\mathbb{R}P^3/G \rightarrow C(\mathbb{R}P^2)$  is induced by the map  $\mathbb{R}P^3 \rightarrow C(\mathbb{R}P^2)$  given by  $(x, y, z) \mapsto ([x, y, z], (x^2 + y^2 + z^2))$  and  $(0, 0, 0) \mapsto p$ . This map is continuous and descends to a map  $\mathbb{R}P^3/G \rightarrow C(\mathbb{R}P^2)$  because negating coordinates does not matter (because all coordinates get squared). So we have a continuous bijection between compact Hausdorff spaces, i.e. a homeomorphism. However,  $C(\mathbb{R}P^2)$  is not a topological manifold (with or without boundary) because any neighborhood  $U$  of  $p$  has  $H_k(U, Up) \cong H_k(C(\mathbb{R}P^2), C(\mathbb{R}P^2)p) \cong H_{k+1}(\mathbb{R}P^2)$  by excision. This proves  $U$  cannot be homeomorphic to  $\mathbb{R}^n$  for any  $n$ .

This example actually shows even more than we expected, because the group which is acting is a Lie group. So quotients of manifolds by Lie groups need not be manifolds. The standard way to fix this is to place more restrictions on the type of action. The key hypothesis (in addition to the action being properly discontinuous) which one often needs is that the action is **free**, which means fixed-point free (i.e. all stabilizer groups are trivial). Under these hypotheses, we have the following general theorem ([1])

**Theorem 4.** Suppose a discrete group  $G$  acts on a  $C^p$  manifold  $M$  in a  $C^p$  way (i.e. all action maps  $\theta_g$  are  $C^p$ ). Suppose the action is free and properly discontinuous. Then there is a unique  $C^p$  structure on the quotient  $M/G$  (taken in  $Top$ ) such that the projection  $\pi : M \rightarrow M/G$  is a local  $C^p$  isomorphism. Moreover,  $M/G$  has the universal property that any  $G$ -equivariant  $C^p$  map out of  $M$  factors uniquely through  $M/G$ .

In the classical setting,  $p$  would be  $\infty$  and this theorem would still hold. When the group  $G$  is a compact Lie group then one can often say more. For instance, the Quotient Manifold Theorem says

**Theorem 5.** *If a Lie group  $G$  acts smoothly, freely, and properly on  $M \in \mathit{Mnfd}$  then  $M/G$  is a topological manifold of dimension equal to  $\dim M - \dim G$  and has a unique smooth structure making  $\pi : M \rightarrow M/G$  a smooth submersion.*

Indeed, this unique smooth structure makes  $M$  into a principal  $G$ -bundle over  $M/G$ . Note that here proper but not properly discontinuous is needed. This is one benefit of working in the Lie group context. Another benefit of working in the Lie group context is that any continuous action of a compact Lie group on a topological manifold is proper. However, it is disappointing that even Lie group actions must be free in order to conclude that the orbit space is a manifold.

In homotopy theory, cofibrant things are built from free things. So if there was a homotopy theory on this category then perhaps one might be able to weaken the hypothesis that the action be free to some kind of cofibrancy hypothesis. This type of weakening appears in my thesis in the context of fairly general model categories  $\mathcal{M}$ . In particular, we call an object  $M \in \mathcal{M}$  Power Cofibrant if a certain technical condition holds which causes  $M$  to become cofibrant in the projective model structure  $\mathcal{M}^G$  on objects with group actions and which causes  $M/G$  to be an object in the underlying category  $\mathcal{M}$ . Basically, this is replacing the idea of “acting freely” with the more general idea of a homotopy colimit equaling a colimit. If one could introduce more homotopy theory to the categories  $\mathit{Mnfd}$  or  $\mathit{LieGp}$  then this notion might lead to a weakening of the theorem above. This goal—along with the realizations that  $\mathit{Mnfd}$  and  $\mathit{LieGp}$  are poorly behaved under basically all types of colimits no matter how nicely we set them up—leads us to try Voevodsky’s approach of embedding  $\mathit{Mnfd}$  and  $\mathit{LieGp}$  into a cocomplete category and working there instead.

**4.2. Embedding into a cocomplete category.** Let  $\mathit{TopMnfd}$  the category of topological manifolds. In [3], Dan Dugger followed Voevodsky’s model with  $\mathit{TopMnfd}$  as opposed to  $\mathit{Sm/k}$  as input. Again using Grothendieck topologies and sheaves, he found the universal model category into which  $\mathit{TopMnfd}$  embeds and proved it’s Quillen equivalent to the model category of topological spaces (Quillen equivalence is the correct notion of equivalence for model categories). This is a bit disappointing because it means all the geometric data is lost in the embedding, so homotopy theorists will have a hard time proving anything new about manifolds using this perspective. One neat corollary to Dugger’s work is that any manifold is the colimit of its atlas, but this doesn’t help us much in understanding which category colimits of manifolds should live.

For smooth manifolds, the correct cocomplete category in which to embed  $\mathit{Mnfd}$  is  $\mathit{Diff}$ . Unlike in the situation for  $\mathit{TopMnfd}$ , we now retain some geometric information in  $\mathit{Diff}$ . So this may be a place where homotopy theory can find non-trivial applications. We describe the objects of  $\mathit{Diff}$ :

**Definition 6.** ([5]) *A **Diffeological Space** is a set  $X$  with a family of maps  $U \rightarrow X$  (called **plots**) for each open  $U \subset \mathbb{R}^n$  and each  $n$ , such that for every open  $U, V \subset \mathbb{R}^n$  we have*

- (1) *Every constant map  $U \rightarrow X$  is a plot*
- (2) *If  $U \rightarrow X$  is a plot and  $V \rightarrow U$  is smooth, then the composition  $V \rightarrow X$  is a plot*
- (3) *If  $U = \bigcup U_\alpha$  and  $U \rightarrow X$  is a map such that every restriction  $U_\alpha \rightarrow X$  is a plot, then  $U \rightarrow X$  is a plot*

The morphisms in  $\mathit{Diff}$  are maps  $X \rightarrow Y$  such that for all plots  $U \rightarrow X$  the composition  $U \rightarrow Y$  is a plot. The category of smooth manifolds embeds as a full subcategory of  $\mathit{Diff}$  where charts are plots and smooth maps of manifolds give maps in  $\mathit{Diff}$ . Furthermore,  $\mathit{Diff}$  is closed under limits and colimits, which is easily seen by noting that the forgetful functor to  $\mathit{Sets}$  preserves both limits and colimits because it has both

left (discrete diffeology) and right (indiscrete diffeology) adjoints. For example, if  $Y$  is a quotient of  $X$  then we can declare  $f : U \rightarrow Y$  to be a plot if for all  $p \in U$  there is a neighborhood  $V$  where  $f$  is of the form  $V \rightarrow X \rightarrow Y$ . Similarly, if  $A$  is a subset of  $X$  then we can declare  $f : U \rightarrow A$  to be a plot if  $U \rightarrow A \rightarrow X$  is a plot.

In some ways  $Diff$  is even nicer than  $Top$ . For instance,  $Diff$  is Cartesian closed, meaning the space of smooth maps between any two diffeological spaces is a diffeological space. This fails for manifolds and fails for topological spaces (unless we restrict attention to compactly generated weak hausdorff spaces and use the compact-open topology on  $C(X, Y)$ ). Furthermore,  $Diff$  is locally presentable but  $Top$  is not (this term means there is a set  $S$  of small objects and every object can be obtained as a colimit of a diagram of objects in  $S$ ). Finally,  $Diff$  admits lots of geometry, e.g. dimension, differential forms, de Rham cohomology, tangent spaces, tangent bundles, etc. So the category  $Diff$  is worth studying in its own right, and also permits the types of constructions (quotients, etc) which homotopy theorists need. The first step towards bringing homotopy theory to this category would be to find a model structure on  $Diff$ . This was mostly done in Enxin Wu's 2012 thesis [12] but has not yet been completed. It's a problem which I might work on at some point if it remains open.

It is worth noting that the category of Lie Groups sits inside the category of smooth manifolds, and also embeds as a full subcategory of  $Diff$ . Furthermore, Lie groups are particularly nice in Wu's proposed model structure on  $Diff$ ; they are fibrant, which means they carry information which is important to the homotopy theory of  $Diff$ . More generally, any homogeneous space is fibrant and one can prove from this fact that all manifolds in  $Diff$  are fibrant, since the way in which  $Mnfd$  embeds in  $Diff$  sends individual manifolds to homogeneous spaces. The proof of this fact is to note that for any  $M$ , the group of self-diffeomorphisms  $G = Diff(M, M)$  is a smooth group. So letting  $H$  be the (smooth) subgroup of diffeomorphisms which fix any point gives  $M \cong G/H$ . This proof shows the strength of  $Diff$ , since the  $G$  above can be less rigid than a Lie group would be in general, but still works for this construction.

**4.3. Orbifolds.** The final way to deal with the lack of colimits in  $Mnfd$  (and in particular the fact that we can't mod out by group actions) is to change the notion of a group action. Since manifolds are defined locally, perhaps we should allow local group actions, which can be different (and even different groups) in each chart. This idea leads to the notion of an orbifold. An **orbifold** is a topological space which is locally a quotient of Euclidean space by the linear action of a finite group. So if one starts with a  $G$ -manifold  $M$  then  $M/G$  is an orbifold, but not all orbifolds arise this way because general orbifolds allow different groups to act in different neighborhoods. Furthermore, even though orbifolds fix the problem that  $Mnfd$  is not closed under passage to coinvariants, the category  $Orbi$  of orbifolds is not cocomplete. So moving from  $Mnfd$  to  $Orbi$  gives some colimits (those related to group actions) but not all (e.g. pushouts and coequalizers can fail).

Because this is a paper on colimits, we shall not spend much more time on orbifolds. We mostly mention them for completeness. To a homotopy theorist, passage from  $Mnfd$  to  $Orbi$  is a nice step, but passage all the way to  $Diff$  is what should be done. So far geometers have found more use for orbifolds than for diffeologic spaces so there are good arguments for studying both categories. In particular, orbifolds let you keep more geometric information than diffeologic spaces and it's often easier to get back to facts about manifolds from facts about orbifolds, but orbifolds do not permit methods from homotopy theory as nicely as diffeologic spaces.

## 5. CONCLUSION

I began this project with the hope of classifying which diagrams in  $Mnfd$  and  $LieGp$  admit colimits in  $Mnfd$ . Unfortunately, even very nice diagrams of very nice manifolds (copies of  $\mathbb{R}$  in fact!) do not admit colimits which are manifolds. Since colimits are so poorly behaved in  $Mnfd$ , one can't even ask about

colimits of Lie groups. Still, many have gotten around these difficulties, e.g. via schemes and diffeologies. Indeed, getting around the lack of colimits was fundamental in Voevodsky's Fields Medal winning work. Perhaps in the future others will find yet more clever ways to deal with these difficulties, or perhaps homotopy theory will become even more general so that the lack of colimits no longer presents an obstacle, e.g. if the work in [12] yields an appropriate  $\infty$ -category. Time will tell, and I for one am excited to see what happens.

#### REFERENCES

- [1] Brian Conrad, "Quotients by Group Actions," preprint, [math.stanford.edu/~conrad/diffgeomPage/handouts/qtmanifold.pdf](http://math.stanford.edu/~conrad/diffgeomPage/handouts/qtmanifold.pdf)
- [2] Jason DeVito, answer to "Proper, smooth action with singular orbit space," 2011, <http://math.stackexchange.com/questions/69263/proper-smooth-action-with-singular-orbit-space>
- [3] Dan Dugger, "Sheaves and Homotopy Theory," preprint, 1998, <http://pages.uoregon.edu/ddugger/cech.html>
- [4] Anton Geraschenko, answer to "Colimits of Schemes," 2010, <http://mathoverflow.net/questions/9961/colimits-of-schemes>
- [5] Patrick Iglesias-Zemmour, *Diffeology*, AMS Mathematical Surveys and Monographs, 2013.
- [6] Saunders MacLane, *Categories for the Working Mathematician*, Springer-Verlag, 1971.
- [7] Andrew Stacey, answer to "Colimits in the category of smooth manifolds," 2010, <http://mathoverflow.net/questions/19116/colimits-in-the-category-of-smooth-manifolds>
- [8] Colin Tan, "Colimits of Manifolds," 2010, <http://mathoverflow.net/questions/38575/colimits-of-manifolds>
- [9] Vladimir Voevodsky, "The Milnor Conjecture," preprint, 1996, K-theory Preprint Archives, <http://www.math.uiuc.edu/K-theory/0170/>
- [10] Vladimir Voevodsky and Fabien Morel, " $A^1$ -homotopy theory of schemes," preprint, 1998, K-theory Preprint Archives, <http://www.math.uiuc.edu/K-theory/0305/>
- [11] Charles Weibel, "A Roadmap of Motivic Homotopy and Homology Theory," in *Axiomatic, enriched and motivic homotopy theory*, 385-392, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer Acad. Publ., Dordrecht, 2004
- [12] Enxin Wu, *A Homotopy Theory for Diffeological Spaces*, PhD Thesis, University of Western Ontario, 2012.