THE SPECTRAL CATEGORY AND VON-NEUMANN REGULAR RINGS

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Abstract. All rings considered are associative with identity and all occurring modules are unital right modules. We denote by Mod R the category of all R-modules.

The spectrum of a ring R is known to be the “set” of isomorphism classes of indecomposable injective right R-modules. When R is right-Noetherian the spectrum describes all injective R-modules, since each injective module is a direct sum of indecomposable submodules. We want to briefly show that for any R (or even for every Grothendieck category), this spectrum can be replaced by the so-called spectral category; one obtains this spectral category by formally inverting all essential monomorphisms. Approaches to such considerations are provided by the work of JOHNSON[6] and UTUMI[8].

As an application one obtains for each module invariant, that the invariant coincides with that which FUCHS[3] introduced under strong assumptions.

1. The Spectral Category of a Grothendieck Category

1.1. Let $\mathfrak{A}$ be a Grothendieck category[1], i.e. an abelian category with exact direct limits and a generator. Exactness of direct limits is equivalent to the following statement:

(*) For each family $(A_\lambda)_{\lambda\in\Lambda}$ of objects of $\mathfrak{A}$

$$B \subset \bigoplus_{\lambda\in\Lambda} A_\lambda \text{ is } B = \sup_{\Gamma}\left(B \cap \bigoplus_{\lambda\in\Gamma} A_\lambda\right)$$

where all finite subsets of Λ factor through Γ.

(Often you see (*) as a special case of AB5)[5]. Suppose conversely that $(A_\lambda)_{\lambda\in\Lambda}$ is an increasing filtered family of subobjects of an object A and A' is another subobject of A. When

$$p : \bigoplus_{\lambda\in\Lambda} A_\lambda \to A$$

[1] Unlike FREYD[2] we require not only that the objects form a set (i.e. the category is small), but also the existence of generators

is the canonical morphism, then:

$$A' \cap \sup_{\lambda\in\Lambda} A_\lambda = p(p^{-1}A') = p\left(\sup_{\Gamma}\left(p^{-1}A' \cap \bigoplus_{\lambda\in\Gamma} A_\lambda\right)\right) = \sup_{\Gamma} p\left(p^{-1}A' \cap \bigoplus_{\lambda\in\Gamma} A_\lambda\right)$$

$$= \sup_{\Gamma}\left(A' \cap p\left(\bigoplus_{\lambda\in\Gamma} A_\lambda\right)\right) = \sup_{\Gamma}\left(A' \cap \sup_{\lambda\in\Gamma} A_\lambda\right) = \sup_{\lambda\in\Lambda}\left(A' \cap A_\lambda\right)$$

where all finite subsets of Λ factor through Γ and the last equation holds because $(A_\lambda)_{\lambda\in\Lambda}$ is an increasing filtration.

1.2. A monomorphism $i : A \to B$ is called essential if the condition $i(A) \cap B' = 0$ for $B' \subset B$ implies $B' = 0$.
We define the spectral category $\mathrm{Spec}\,\mathfrak{A}$ of $\mathfrak{A}$: the spectral category has the same objects as $\mathfrak{A}$. For A and B objects of $\mathrm{Spec}\,\mathfrak{A}$,

$$(\mathrm{Spec}\,\mathfrak{A})(A,B) = \varinjlim \mathfrak{A}(A',B)$$ [2]

where the direct limit is taken over all essential subobjects $A' \subset A$. Elements $f \in \mathrm{Spec}\,\mathfrak{A}(A,B)$ and $g \in \mathrm{Spec}\,\mathfrak{A}(B,C)$ are now determined by the diagrams:

$$\begin{tikzcd} A & B \\ A' \arrow[u, hook] \arrow[ur, "f'"] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} B & C \\ B' \arrow[u, hook] \arrow[ur, "g'"'] & \end{tikzcd}$$

where A' and B' are essential in A and B. Then $f^{-1}(B')$ is essential in A and gf is defined via the diagram:

$$\begin{tikzcd} A & C \\ f'^{-1}(B') \arrow[u, hook] \arrow[ur, "g'(f'|f'^{-1}(B'))"'] & \end{tikzcd}$$

The categories $\mathfrak{A}$ and $\mathrm{Spec}\,\mathfrak{A}$ become connected by the canonical functor $P : \mathfrak{A} \to \mathrm{Spec}\,\mathfrak{A}$ which is the identity on objects and which has $\mathfrak{A}(A,B)$ as the natural image in

$$\varinjlim_{A'} \mathfrak{A}(A',B)$$

1.3.

Theorem. *Suppose* $\mathfrak{A}$ *is a Grothendieck category. Then* $\mathrm{Spec}\,\mathfrak{A}$ *is a Grothendieck category in which each morphism decomposes (i.e. Ker* f *and Im* f *are direct summands in the source and target of* f*).*

Proof. It is clear that $\mathrm{Spec}\,\mathfrak{A}$ is an additive category and P is an additive functor. Let $f \in (\mathrm{Spec}\,\mathfrak{A})(A,B)$ be a morphism as considered in 1.2. Let A_1 be the complement of $\mathrm{Ker}\, f'$ in A' (i.e. $A_1 \cap \mathrm{Ker} f' = 0$ and A_1 is maximal with respect to this property) and B_1 the complement

[2] If A and B are objects in the category $\mathfrak{A}$ we write $\mathfrak{A}(A,B)$ instead of $\mathrm{Hom}_{\mathfrak{A}}(A,B)$

of $f'(A_1)$ in B. This results in a commutative diagram

$$\begin{tikzcd}
Kerf' \oplus A_1 \arrow[r, "f''"] \arrow[d, "i"'] & f'(A_1) \oplus B \arrow[d, "k"'] \\
A' \arrow[r, "f'"] \arrow[d, "j"'] & B \\
A &
\end{tikzcd}$$

where f'' is induced by f' and decomposes via the essential inclusions i, j, k. Since Pf'' is isomorphic to f and decomposes, f also decomposes. In particular, Spec $\mathfrak{A}$ is an abelian category and P is left exact.

Furthermore Spec $\mathfrak{A}$ has infinite direct sums and P commutes with direct sums, since a direct sum of essential monomorphisms is an essential monomorphism in the Grothendieck category $\mathfrak{A}$.

In general, P commutes with intersections (because of left-exactness) and with a supremum of a filtered system of subobjects (analogous to the proof for direct sums). Therefore (*) from 1.1 holds in Spec $\mathfrak{A}$.

Finally, PU is a generator in Spec $\mathfrak{A}$ if U is a generator in $\mathfrak{A}$. □

1.4. The canonical functor P is an isomorphism iff every morphism in $\mathfrak{A}$ is decomposable. A Grothendieck category in which each morphism decomposes is therefore called a spectral category.

1.5.

Theorem. *For objects $A, B \in \mathfrak{A}$ the following are equivalent*

i) PA is isomorphic to PB

ii) There is an object $C \in \mathfrak{A}$ and essential monomorphisms $i : C \to A$ and $j : C \to B$

iii) A and B are isomorphic to their injective hulls

The proof is clear

Therefore there is a 1-1 correspondence between the isomorphism classes of injective objects of $\mathfrak{A}$ and the isomorphism classes of all objects of Spec $\mathfrak{A}$. An object $A \in \mathfrak{A}$ is co-irreducible iff PA is simple. At the same time, an object A is coirreducible if it's nonzero and each different subobject is essential in A.

2. Characterizing the Spectral Category with the help of regular rings

2.1.

Theorem. *Let $\mathfrak{S}$ be a spectral category with generator U and $R = \mathfrak{S}(U, U)$. Then R is regular*[3] *and is an injective R-module. The functor*

$$S \xrightarrow{F} \mathfrak{S}(U, S)$$

[3] R regular means each principal ideal is a direct summand in R

is an equivalence of $\mathfrak{S}$ *onto the full subcategory of Mod* R *consisting of direct summands of powers of* R*.*

Proof. That the functor F is an equivalence onto the full subcategory follows from [7]. From [7] it also follows that infinite direct products are in $\mathfrak{S}$ because F is an equivalence to the induced quotient category of Mod R (the existence of inverse limits follows from the existence of direct limits, and is also true under much more general assumptions (see e.g. [1]).) Since each morphism in $\mathfrak{S}$ decomposes and U is a generator, it is also an injective cogenerator. To each $S \in \mathfrak{S}$ there is thus a monomorphism i from S into the product

$$\prod_{\lambda} U_\lambda$$

for $U_\lambda \cong U$. Since i decomposes, Fi also decomposes, i.e. FS is a direct summand of

$$F(\prod_{\lambda} U_\lambda) \cong \prod_{\lambda} F(U_\lambda)$$

however, $F(U_\lambda)$ is isomorphic to R.

Suppose conversely that p is an idempotent endomorphism of a power $R^{\aleph}$ of R and $M = p(R^{\aleph})$. Then p is of the form Fq for $q \in \mathfrak{S}(U^{\aleph}, U^{\aleph})$. Therefore $M = \mathrm{Ker}(1-p) = F(\mathrm{Ker}(1-q))$. We now finally show that R is self injective: Let $I \subset R$ be a right ideal. We want to show that the natural image of R in $\mathrm{Hom}_R(I, R)$ is surjective. There are the following identifications:

$$\mathrm{Hom}_R(I,R) = \mathrm{Hom}_R(\varinjlim M, R) = \varprojlim \mathrm{Hom}_R(M,R) = \varprojlim \mathfrak{S}(V,U) = \mathfrak{S}(\varinjlim V, U)$$

where M is a finitely generated submodule of I, hence a direct summand of R, hence it has the form FV, $V \subset U$. But

$$\varinjlim V$$

is a direct summand in U; therefore the natural image

$$\mathrm{Hom}(U,U) = R \to \mathfrak{S}(\varinjlim V, U) = \mathrm{Hom}_R(I,R)$$

is surjective □

2.2.

Theorem. *Let* R *be a regular, self-injective ring. Then the full subcategory of Mod* R *of direct summands of powers of* R *is a spectral category.*

Proof. Let $\mathfrak{C}$ be the localizing subcategory [4, pg. 377] of all modules C of Mod R with $\mathrm{Hom}_R(C, R) = 0$.

A right ideal $I \subset R$ is essential iff $R/I \in \mathfrak{C}$. Let $R/I \in \mathfrak{C}$ and $e \in R$ with $I \cap eR = 0$. Given any morphism $f : eR \to R$, form a monomorphism from eR to R/I by continuing f to R/I

hence $\mathrm{Hom}_R(eR, R) = 0$ so $eR = 0$ as eR is a direct summand of R.

Conversely suppose I is essential in R, $f : R/I \to R$ is a homomorphism, and g is the composition of f with projection from R to R/I. The image $g(R)$ is cyclic, hence projective, therefore Ker g is a direct summand in R and is essential, hence Ker $g = R$ and $f = 0$.

We now know that Mod $R/\mathfrak{C}$ is a Grothendieck category [4, pg. 378] and that the canonical functor from Mod R to Mod $R/\mathfrak{C}$ possesses a right adjoint functor S. This S induces an equivalence of Mod $R/\mathfrak{C}$ to the full subcategory of all $\mathfrak{C}$-closed objects of Mod R. This means a module M is $\mathfrak{C}$-closed when 0 is the unique submodule of M in $\mathfrak{C}$ and when $R/I \in \mathfrak{C}$ implies every morphism $f : I \to M$ extends to R. In particular, each injective module is $\mathfrak{C}$-closed when 0 is the unique submodule in $\mathfrak{C}$. The direct summands of powers of R are also $\mathfrak{C}$-closed.

Conversely let M be a $\mathfrak{C}$-closed module. Let I be any right ideal and let I' be the complement of I, so each homomorphism $f : I \to M$ extends to $I \oplus I'$ and also to R. Thus $I \oplus I'$ is essential in R and therefore $R/I \oplus I'$ is in $\mathfrak{C}$. This means that M is injective, hence in particular that the $\mathfrak{C}$-closed modules form a spectral category.

For $f \in \mathrm{Hom}_R(M, R)$, the R_f are also copies of R. The canonical image $\phi(M)$ of M in

$$\prod_f R_f$$

is injective because Ker ϕ belongs to $\mathfrak{C}$. Hence M is a direct summand in $\prod R_f$. The $\mathfrak{C}$-closed modules are hence exactly direct summands of powers of R. □

3. Applications

3.1. $\mathfrak{S}$ is a spectral category so each object S of $\mathfrak{S}$ is a direct sum of its base So S, i.e. a sum of simple subobjects of S, and its radical Ra S, i.e. the intersection of maximal subobjects of S. We denote by $\mathfrak{S}_d$ (d for discrete) or $\mathfrak{S}_k$ (k for continuous) the full subcategory of all objects whose radical or base is 0, so the functor $S \to$ (So S, Ra S) is an equivalence of $\mathfrak{S}$ with $\mathfrak{S}_d \times \mathfrak{S}_k$. If $\mathfrak{E}$ is another representative system of isomorphism classes of simple objects in $\mathfrak{S}$ then it's known that the functor $S \to \mathfrak{S}(E, S)_{E \in \mathfrak{E}}$ is also an equivalence of the semi-simple category $\mathfrak{S}_d$ and the product category

$$\prod_{E \in \mathfrak{E}} \mathrm{Mod}\, \mathfrak{S}(E, E)$$

the category of vector spaces over the skew field $\mathfrak{S}(E, E)$

From 2.2 and the previous comments we obtain

Theorem. *Each regular, self-injective ring R is the ring direct product of its base So S and the finite radical $\mathrm{Ra}_e R$ (this is the intersection of all maximal direct summands of R). This $\mathrm{Ra}_e R$ is a* *regular, self-injective ring whose base vanishes.*

3.2. For each $E \in \mathfrak{E}$ and $S \in \mathfrak{S}$ let $r_E(S)$ be the well-defined dimension of $\mathfrak{S}(E,S) = \mathfrak{S}(E, \mathrm{So}\, S)$ over $\mathfrak{S}(E,E)$. Often this is characterized as the magnitude $r_E(S)$, $E \in \mathfrak{E}$ of the basis of S up to isomorphism. The function r_E has the following properties:

i) If S and T are isomorphic, then $r_E(S) = r_E(T)$

ii) If $S = \bigoplus_\lambda S_\lambda$ then $r_E(S) = \sum_\lambda r_E(S_\lambda)$

iii) If F is simple then

$$r_E(F) = \begin{cases} 1 \text{ for } F \cong E \\ 0 \text{ otherwise} \end{cases}$$

iv) If $\mathrm{So}\, S = 0$ then $r_E(S) = 0$

It is clear that r_E is the unique function from the objects of $\mathfrak{S}$ to the cardinal numbers with properties i) through iv).

3.3. Suppose now $\mathfrak{A}$ is any Grothendieck category with spectral category Spec $\mathfrak{A}$, $P : \mathfrak{A} \to \mathrm{Spec}\, \mathfrak{A}$ the canonical functor, and $\mathfrak{F}$ a representative system of isomorphism classes of indecomposable injective objects in $\mathfrak{A}$. Then $P\mathfrak{F}(= \mathfrak{F})$ is a representative system of isomorphism classes of simple objects of Spec $\mathfrak{A}$.
Carrying over the results of 3.1 and 3.2 for $\mathrm{Spec}\, \mathfrak{A}$ and $\mathfrak{A}$ one obtains the following result:

Theorem. *Suppose $(I_\gamma)_{\gamma \in \Gamma}$ and $(J_\lambda)_{\lambda \in \Lambda}$ are direct families of indecomposable injective objects, A is an object, and $i : \bigoplus_\gamma I_\gamma \to A$ and $j : \bigoplus_\lambda J_\lambda \to A$ are essential monomorphisms in $\mathfrak{A}$. Then there is a bijection $b : \Gamma \to \Lambda$ with $J_{b(\gamma)} \cong I_\lambda$ for all $\gamma \in \Gamma$.*

Furthermore, for all $I \in \mathfrak{F}$ and $A \in \mathfrak{A}$, $r_I(A) := r_{PI}(PA) =$ dimension of $(\mathrm{Spec}\, \mathfrak{A})(PI, PA)$ over $(\mathrm{Spec}\, \mathfrak{A})(PI, PI)$. NOTE: original paper had "$I \in J$", a typo
The function r_I has the following properties:

i) If $f : A \to B$ is an essential monomorphism then $r_I(A) = r_I(B)$

ii) If $A = \bigoplus_\lambda A_\lambda$ then $r_I(A) = \sum_\lambda r_I(A_\lambda)$

iii) If A is coirreducible then

$$r_I(A) = \begin{cases} 1 \text{ if I is isomorphic to the injective hull of } A \\ 0 \text{ otherwise} \end{cases}$$

iv) If A contains no coirreducible subobject then $r_I(A) = 0$

Via the properties i) through iv), r_I becomes uniquely defined. The number $r_I(A)$ is called the I-rank of A. It can be calculated as follows: let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of coirreducible subobjects of A, maximal with respect to the property

$$\sup_\lambda A_\lambda = \bigoplus_\lambda A_\lambda$$

(Zorn's Lemma). Then $r_I(A)$ is the cardinality of the set of all A_λ whose injective hulls are isomorphic to I. Specifically, if U is a generator of $\mathfrak{A}$ then A is U-cyclic, i.e. a choice of epimorphic image of U.

As a corollary one obtains e.g: suppose $(A_\gamma)_{\gamma\in\Gamma}$ and $(A_\lambda)_{\lambda\in\Lambda}$ are families of copies of an object A in $\mathfrak{A}$. Then

$$\bigoplus_\gamma A_\gamma \cong \bigoplus_\lambda A_\lambda$$

and there is an $I \in \mathfrak{F}$ with $0 < R_I(A) < \aleph_0$ so Γ and Λ have the same cardinality.

Using the preceding on modules over a ring, the result that basis length of free modules is unique follows. Furthermore, this follows without any of the assumptions of Fuchs[3] on the ring.

4. References

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