# Phase space logics for analysis of physical models of computation

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## Introduction

Phase space logics were developed to provide examples of derived logics. The motivation for this was an issue concerning quantum logic: Is quantum logic more fundamental than classical Boolean logic? Hilary Putnam argued that the relationship between Boolean logic and quantum logic was closely analogous to that between non-Euclidean geometry and Euclidean geometry: in each case, the latter system is merely an approximation of the more fundamental former system. In [2] the argument was made that, instead of being more fundamental, the von Neumann - Birkhoff quantum logic has a purely derived status. To be more precise, quantum logic is derived from the measurement theory of quantum mechanics and the mathematical structure of the Hilbert space, which is used to model a given quantum system.

Phase space logics were used as further examples of derived logics. Phase space logics are derived logics which can be used to analyze *classical* mechanical systems. To derive these logics, a rudimentary measurement theory is developed for classical mechanics and is used to analyze the topological structure of a mechanical system's phase space. The surprising result is that the logics derived from this process are non-Boolean. Thus the phase space logics provide further examples of non-Boolean systems which can be used to analyze physical systems when one is careful to describe what constitutes measurement.

It is this attention to the nature of measurement which motivates the current study. In any computation the measurement of some physical system is necessary. Voltage levels, strength of magnetized domains, and the location of billiard balls are just a few examples of systems whose measurement constitute

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a computation. If measurement is of the essence in computation, then one is lead to ask: What logic is appropriate for the analysis of computation? In this paper billiard ball models of computation will be analyzed by three varieties of phase space logics: bivalent open phase space logic, three-valued open phase space logic, and twin open set phase space logic. The analysis of billiard ball computation by phase space logic is motivated by noting that phase space logics were developed to reflect the fact that measurements of position are never infinitely precise. It is the measurement of positions of billiard balls that is computation in these models.

The paper is organized as follows. Section I presents the general notion of a derived logic. In Section II, the properties of bivalent open phase space logic are discussed. This presentation is a summary of results presented in greater depth in [2]. This provides an example of the ideas presented in Section I. In section III, bivalent open phase space logic is applied to the analysis of a physical model of computation: the billiard ball model. Section IV presents the properties of three-valued open phase space logic. This presentation is a summary of results presented in greater depth in [3].Section V presents the application of three-valued open phase space logis to the billiard ball model. In Section VI we present the properties of a third logic: the twin open set phase space logic. As this logic has not been previously described, considerable space will be devoted to this presentation. The results presented in Section VII parallel those presented in Sections III and V; this time we analize billiard ball computation using twin open sep phase space logic. Section VIII provides a brief summary and indication of possible extension of the present work.

## I. Derived Logics

Let us consider a simple example of a derived logic in order to illustrate the concept. We take our system to be a point particle which is constrained to lie somewhere along a line. $\mathbb{R}$ is the mathematical space of configurations for this system. We operate under the (unphysical) assumption that measurements of the particle's location are infinitely precise. Thus, we will be able to identify measurements with arbitrary subsets of $\mathbb{R}$—for example, infinite precision allows us to experimentally determine whether the particle's coordinate is a rational number or not. A measurement in this dream-land consists of the unambiguous determination of whether the system's configuration lies within a particular set or not.

In this system, propositions about the particle would be of the form: "The particle is in $X$," where $X$ is a subset of $\mathbb{R}$ representing positions. In this case, propositions about the particle will also be identified with subsets of $\mathbb{R}$. This is because any two distinct subsets of $\mathbb{R}$ may be distinguished from each other by some measurement. Indeed, given sets $A$ and $B$ in $\mathbb{R}$ the measurement (subset) $A \cap B^c$ (where the superscript "c" stands for complementation) will distinguish between the sets.

As we wish for logical operators to map sets of propositions to a proposition, we will identify logical operators with set operations as follows:

$$A \vee B = A \cup B \tag{1}$$
$$A \wedge B = A \cap B \tag{2}$$
$$\neg A = A^c \tag{3}$$
$$A \rightarrow B = A^c \cup B. \tag{4}$$

The expression on the right-hand side of each equation treats the objects $A$ and $B$ in their ordinary set theoretic guises; each expression yielding the definition of the logical operator acting on $A$ and $B$, now treated as propositions, on the left-hand side.

These definitions accord with our intuition concerning the truth of a proposition as determined by a measurement. For example, if a measurement yields true (respectively, false) for $A$ and true (respectively, false) for $B$, the same measurement will yield true for $A \cap B$ (respectively, false for $A \cup B$). Similarly, $\neg A$ is determined to be true by a measurement (respectively, false) if and only if $A$ is determined to be false (respectively, true) by the same measurement.

Given this mathematical state space and this theory of measurement, the derived logic for this system will be ordinary Boolean logic. This follows from the fact that the calculus of set operations is a Boolean lattice. It is interesting to note that a theory of measurement, which is only slightly more sophisticated than the one considered here, will yield non-Boolean logics.

The main thesis of [2] is that nonstandard derived logics are possible for a wider class of physical theories than quantum mechanics. To illustrate this fact, two nonstandard logics were derived for classical mechanical systems. The nonstandard features of von Neumann - Birkhoff logic flow from the Hilbert space (more specifically, the vector space) structure of quantum mechanics. The nonstandard features of the phase space logics derived in [2] flow from the topological structure of the phase space associated with a physical system by classical dynamics. More specifically, the open phase space logic is derived from the structure of open subsets of the phase space while closed phase space logic is derived from the structure of closed subsets of the phase space. These logics are motivated by a rudimentary theory of measurement, in which a given measurement outcome localizes the state of the system to an open set in phase space rather than an idealized point. Subsets of phase space which cannot be distinguished (in one sense or another) by such measurements must be identified with the same proposition in the derived logic.

## II. Open Phase Space Logic

In [2] two derived logics for classical mechanical systems were described: the open phase space logic and the closed phase space logic. As *tertium non datur* does not hold in open phase space logic (see Theorem 4 below) it is a natural candidate for a three-valued interpretation. In this section we describe those properties of the open phase space logic which deal with the structure of the propositions in the logic. As these are the same for both the bivalent and three-valued version of open phase space logic, we will not distinguish between the two in the present section. In Section III. we will use the bivalent variant of open phase space logic to analyze billiard ball models of computing.

We will present the proofs of only part of Theorems 2 and 3. These are presented here merely to provide the flavor of the techniques of proof for these systems. A more complete presentation of the proofs can be found in [2].

Let $P$ be a proposition; we recall from [2] that this means that $P$ is an equivalence class of sets. Distinct sets are in the same equivalence class $P$ if and only if they have the same interior, as sets with the same interior will be verified by the same set of measurements (which are to be identified with open sets, as noted in the previous section). The properties of these propositions are determined by the topological properties of subsets of the phase space. Suppose $V$ and $W$ are arbitrary subsets of a topological space $X$. Then it is true that

$$\overline{\overline{V}} = \overline{V} \tag{5}$$
$$\mathrm{int}\,(\mathrm{int}\,V) = \mathrm{int}\,V \tag{6}$$
$$\overline{V} \cap \overline{W} \supseteq \overline{V \cap W} \tag{7}$$
$$\overline{V} \cup \overline{W} = \overline{V \cup W} \tag{8}$$
$$\mathrm{int}\,V \cup \mathrm{int}\,W \subseteq \mathrm{int}\,(V \cup W) \tag{9}$$
$$\mathrm{int}\,V \cap \mathrm{int}\,W = \mathrm{int}\,(V \cap W) \tag{10}$$

($\overline{V}$ and $\mathrm{int}\,V$ denote the closure and interior of the set $V$, respectively.) Proofs of the above facts can be found in any introductory topology text [10].

As we stated earlier, we wish to find a single alternative proposition to $P$; this single alternative proposition will serve as the negation of $P$. The alternative which is readily suggested is $[\mathrm{int}\,(P_i{}^c)]$ where $P_i$ is any element of $P$. This choice of the alternative proposition motivates our definitions of the three-valued operations corresponding to "or", "and", and "not". These operators are defined as follows:

**Definition 1:** Let $P$ and $Q$ be propositions. That is, $P$ is the equivalence class $[P_i]$ and $Q$ is the equivalence class $[Q_j]$. Then we define the logical operators "$\neg$" ("not"),"$\wedge$"("and"), and "$\vee$" ("or") as follows:

$$\neg P = [(\text{int}\, P_i)^c] \tag{11}$$
$$P \wedge Q = [(\text{int}\, P_i) \cap (\text{int}\, Q_j)] \tag{12}$$
$$P \vee Q = [(\text{int}\, P_i) \cup (\text{int}\, Q_j)]. \tag{13}$$

These definitions are all based on the notion that a proposition is to be identified with a substructure of the mathematical state space, in this case, an equivalence class of sets.

**Theorem 2.** The operations $\vee$ and $\wedge$ are both commutative and associative. Furthermore, $\vee$ distributes over $\wedge$, and $\wedge$ distributes over $\vee$.

**Proof.** We will omit the proofs of commutativity and associativity, and give only the proof for the distributivity of $\wedge$ over $\vee$. Other parts of the proof are straightforward.

Suppose $A$, $B$, and $C$ are propositions in open phase space logic. Then

$$A \wedge (B \vee C) = [\text{int}\, A_\mu] \wedge [\text{int}\, B_\nu \cup \text{int}\, C_\eta] \tag{14}$$
$$= [\text{int}\, A_\mu \cap \text{int}\, (\text{int}\, B_\nu \cup \text{int}\, C_\eta)]. \tag{15}$$

Referring to the topological properties listed in (1)–(6) we see that

$$A \wedge (B \vee C) = [\text{int}\, A_\mu \cap (\text{int}\, B_\nu \cup \text{int}\, C_\eta)] \tag{16}$$
$$= [(\text{int}\, A_\mu \cap \text{int}\, B_\nu) \cup (\text{int}\, A_\mu \cap \text{int}\, C_\eta)]. \tag{17}$$

The intersection of two open sets is open, so

$$\text{int}\, V \;\cap\; \text{int}\, W = \text{int}\, (\text{int}\, V \;\cap\; \text{int}\, W). \tag{18}$$

Thus,

$$A \wedge (B \vee C) = [\text{int}\, (\text{int}\, A_\mu \cap \text{int}\, B_\nu) \cup \text{int}\, (\text{int}\, A_\mu \cap \text{int}\, C_\eta)] \tag{19}$$
$$= [\text{int}\, A_\mu \cap \text{int}\, B_\nu] \vee [\text{int}\, A_\mu \cap \text{int}\, C_\eta] \tag{20}$$
$$= (A \wedge B) \vee (A \wedge C). \tag{21}$$

Therefore, $\wedge$ distributes over $\vee$ in open phase space logic. The proof that $\vee$ distributes over $\wedge$ is similar. □

**Theorem 3.** For a proposition $P$, $P \sqsubseteq \neg\neg P$

**Remark:** Recall from [2] that $P \sqsubseteq Q$ means that for every set $P_i$ in the equivalence class $P$ there is a set $Q_j$ in $Q$ such that $P_i \subseteq Q_j$.

**Proof.** Let $P$ be a proposition; i.e.,

$$\begin{aligned} P &= [P_i] && (22) \\ &= [\text{int}\, P_i]. && (23) \end{aligned}$$

For any set $S$, it is the case that $\text{int}\, S \subseteq \text{int}\,((\text{int}\,(S^c))^c)$. This can be seen as follows: Let $x \in \text{int}\, S$. Then there is a neighborhood of x, denoted by $U_x$, such that $U_x \subseteq \text{int}\, S \subseteq S$. Hence, $U_x \cap S^c = \emptyset$ and so $U_x \cap \text{int}\,(S^c) = \emptyset$. Thus $U_x \subseteq (\text{int}\,(S^c))^c$ so that $x \in \text{int}\,((\text{int}\,(S^c))^c)$ and we have $\text{int}\, S \subseteq \text{int}\,((\text{int}\,(S^c))^c)$ as claimed. Thus, we may continue the proof of the theorem:

$$\begin{aligned} P &\sqsubseteq [\text{int}\,((\text{int}\,(P_i{}^c))^c)] && (24) \\ &= \neg[(\text{int}\,(P_i{}^c))] && (25) \\ &= \neg\neg[P_i] && (26) \\ &= \neg\neg P. && (27) \end{aligned}$$

□

**Theorem 4.** In open phase space logic, $A \wedge \neg A = 0$ ("0" here denotes the equivalence class which contains the empty set, which is the never verified, or always false, proposition) for all propositions $A$ ( thus, in the open phase space logic there is a law of non-contradiction). On the other hand, $A \vee \neg A \neq 1$ for some topological spaces; i.e.,no *tertium non datur* ("1" here denotes the equivalence class which contains the entire space, which is the always verified, or always true, proposition).

**Theorem 5.** For propositions $A$ and $B$ in the open phase space logic,

$$\begin{aligned} &\neg(A \vee B) = \neg A \wedge \neg B && (28) \\ &\neg A \vee \neg B \sqsubseteq \neg(A \wedge B). && (29) \end{aligned}$$

In the second relation, equality need not hold.

The proofs of these facts are straightforward. The two relations work out differently because of the asymmetry between set union and intersection with respect to the interior operation. That is, by the topological properties (1)–(6), the interior of the intersection of two sets is equal to the intersection of their interiors; however, in general the union of their interiors merely contains the interior of the union of those sets.

It is interesting to note that the open phase space logic is Boolean if the underlying phase space has a discrete topology. (The phase spaces of ordinary classical dynamical systems, of course, do not.) Thus, the topological structure of the state space governs the character of the derived logic.

We reiterate that these properties are independent of whether the statements are interpreted using a bivalent ($P$ is verified or not verified) or a three-valued interpretation. For a discussion of the bivalent interpretation of open phase space logic see [2]. The description of the three-valued interpretation follows in Section IV.

## III. Bivalent Open Phase Space Logic and Billiard Ball Computation

As was mentioned in the introductory section, the *derived* in the term "derived logic" refers to the fact that these logical structures are derived from a theory of measurement. In the case of the classical phase space logics, that theory is quite simple; it is embodied in the following set of axioms [2]:

(i) Observables are continuous phase-space functions.
(ii) Values of observables cannot be determined with arbitrarily but not infinitely high precision.
(iii) Only finitely many measurement outcomes are available.

The bivalent open pase space logic was used to analize several physical systems in [2]. In this paper, we consider another physical system: a computer

The mathematics of computing machinery, in particular for current digital computers, has its foundation in boolean algebra. It is a common exercise in beginning computer courses to show that both the propositional logic and collections of certain circuitry together with ways of combining electronic gates form boolean algebras. This fact yields a major advantage to those who need to reason about computing circuits.

However, the use of boolean (standard) logic for reasoning about circuitry ignores questions of measurement. One examines input and output paths of circuits from the point of view of whether high or low voltage is on a particular path. Such an approach necessitates that one assume there is an exact measurement for these values. Of course, our understanding of physics makes it clear that such exactness is not the reality. For this reason it seems reasonable to apply logic that is based on physical measurement to models of computation.

Here, we consider some nonstandard logics with the ideas of seeing what kinds of statements can be expressed about computation models. We are especially interested to see what can be said that is not possible to say using traditional logic. We note that these nonstandard logics do not form boolean algebras, and we propose that in future work we plan to formalize the kind of mathematical system these nonstandard logics take and to make some comparisons with traditional boolean logic.

In the analysis of billiard ball gates, the measurements with which we will be concerned are locations of the balls. In order to simplify the analysis, we will limit ourselves to locating the balls in space rather than the more correct spacetime. Strictly speaking, Axiom 2 must be modified to reflect the fact that, given a particular machine, the precision is finitely limited; i.e., a given machine will have a given resolution.

Let us consider the billiard ball interaction gate shown in Figure 1. The measurement to be considered is whether or not a ball is located at positions labelled $y_1, y_2, y_3$, and $y_4$. In the naive view, one considers these locations to be points and the measurement is to look at the point and see if there is a ball at that point or not.

The situation is actually more complicated. As measurements have only fixed resolution (let us call this resolution $\delta$) a measurement is performed by looking at open intervals about each of the points. How big should these intervals be? The answer to that question depends upon how much error one is willing to tolerate. Hence the result of a measurement depends upon a *tolerance*. the tolerance (let us denote it by $\epsilon$) will fix the size of the intervals about the points $y_1$ to $y_4$ as shown in Figure 2.

We will say the result of the computation by the gate is as follows: proposition " A and B" is verified if ball A is located in interval $y_1$ and ball B is located in interval $y_4$. We should be precise in saying what constitutes the situation "a ball $x$ is located in interval $y$". It is simply this: a measurement $m$ is an interval of width *delta*, we say a ball is located in interval $y$ if $m \subset y$. Complementarily, we say that "ball $x$ is not in $y$" if $m \subset y^c$ (the superscript c denote set complementation).

Thus, the gate will return the result "A and B is verified" if and only if $m_A \subset (y_1 - \epsilon, y_1 + \epsilon)$ and $m_B \subset (y_4 - \epsilon, y_4 + \epsilon)$, where $m_J$ denotes the measurement for ball J (J = A,B). Similarly, the gate will return a value of "B and not A is verified" if and only if $m_B \subset (y_2 - \epsilon, y_2 + \epsilon)$ and $m_A \not\subset Y$ where $Y$ is the entire right hand wall of the gate. The gate will return a value of "A and not B is verified" if and only if $m_A \subset (y_2 - \epsilon, y_2 + \epsilon)$ and $m_B \not\subset Y$.

Thus, one will not set the tolerance $\epsilon$ to be less than the resolution $\delta$; otherwise, the gate will never return a value of verified for any of the four possible outputs of the gate. This does not mean that such a gate would always produce a value of unverified. From the discussion in Section I, we recall that in open phase space logic, a proposition $P$ is assigned the value of "unverified" by a measurement $m$ if and only if $m \subset int[(P_0)^c]$. That is, $m$ assigns "unverified" to $P$ if and only if $m$ is contained entirely inside the complement of any representative of $P$. But this is not the only way in which a measurement can fail to assign a value of "verified" to a proposition.

It is quite possible that a measurement $m$ is such that $m \not\subset P_0$ and also that $m \not\subset int(P_0)^c$. For example, in the interaction gate under consideration here, $m$ with resolution $\delta$ might be the interval $(y_1+\epsilon-\delta/2, y_1+\epsilon+\delta/2)$ (see Figure 3.). Thus the proposition "$A$ and $B$ " is assigned neither a value of "verified" nor a value of "unverified". In fact, in the bivalent open phase space such a measurement fails to assign any value to the proposition "$A$ and $B$". This failure to assign any value motivates using three-valued logics in the analysis of billiard ball computers.

## IV. Three-Valued Open Phase Space Logic

In addition to the considerations noted at the end of the last section it has been noted [2,3] that the law of *tertium non datur* does not hold in open phase space logic. The possibility that the law of *tertium non datur* is not a tautology has previously been a motivation for the development of three-valued logics as possible alternatives to standard Boolean logic (cf. [5], [6], [7], [8], [9]). Thus, the possibility of a three-valued logic for classical mechanical systems is suggested. One benefit of developing a three valued *derived* logic is that we may hope to thereby avoid the ambiguity that often attends the selection of three-valued connectives. As pointed out in [7], twelve distinct "negations" are possible for a three-valued logic. The choice among these is to some degree *ad hoc*. Indeed, in the three-valued system he developed for quantum mechanics, Reichenbach ([6]) chose to include three different negations. As we will show, the derivation of a three-valued logic from the mathematical structure of the state space eliminates some of the *ad hoc* features of the choice of connectives.

As we have seen, when a proposition $P$ is not verified, it does not follow that the negation of P is verified (i.e., "$P$ being not verified" is not equivalent to "$\neg P$ being verified"). Similarly, if the negation of $P$ is not verified we cannot conclude that $P$ is verified. In other words, it is possible that a proposition and its negation may both fail to be verified. This reflects that *tertium non datur* does not hold in open phase space logic; i.e., $P \vee \neg P \neq 1$. As the canonical representatives of $P$ and $\neg P$ do not cover the entire space, one can reasonably expect there to be another alternative. It is the existence of this third alternative that indicates that a three-valued interpretation may be useful for open phase space logic.

There are two points worth noting: First, when dealing with a three-valued logic one must take some care with defining the negation of a proposition. This contrasts with the situation one finds in a two-valued logic. In the two-valued case, only one negation is possible but in a three-valued logic, the existence of more than one alternative value for a given proposition implies that more than one negation is possible. In the present development, as our analysis uses a derived logic, we are able to choose one negation. This choice of one negation is possible because, in a derived logic, logical connectives are associated with substructures of the mathematical state space modeling the physical system.

The following definitions will aid in establishing a three-valued interpretation for open phase space logic.

**Definition 1:** For a set S in the topological space X, we define the *boundary* of S to be the set $\partial S = \overline{S} \cap \overline{S^c}$. A point x is a *boundary point* of $S$ if $x \in \partial S$.

Note: this definition implies that, for a given set, the interior and the boundary of that set are disjoint. Also, for a given set $A$, it is the case that $\overline{A} = int(A) \cup \partial(A)$.

**Definition 2:** Let $S$ be a set in the topological space $X$. Then a point y on the boundary of S is an *adherent point* if and only if every neighborhood of y contains a point of the interior of S. The set of all the adherent points of $S$ is denoted by $adh(S)$.

A few words are in order concerning the idea of an adherent point. Consider a set $P_i$ as an element of the proposition $P$. Roughly speaking, the adherent points of $P_i$ are those boundary points that cannot be "pried away from $P_i$" by any measurement. That is to say, $x \in \partial P_i$ is an adherent point of $P_i$ if, for any open set $m$, (such as those representing measurements) containing $x$ is such that $m \cap P_i$ has a non-empty interior. Stated yet another way: $x \in \partial P_i$ is an adherent point if any measurement that contains $x$, must also contain some of the interior of $P_i$.

Theorems 3 through 5 provide properties of adherent points that are useful in describing the three-valued open phase space logic. Complete proofs of these facts can be found in [3].

**Theorem 3.** If $S$ is an open set, then the set of boundary points of $S$ is equal to the set of adherent points of $S$; i.e. $\partial S = adh(S)$.

**Theorem 4.** Let $[R] = [S]$ (i.e., let int (R) = int (S)), then $adh(R) = adh(S)$.

**Remark:** It should be noted that Theorem 4 allows us to talk of the *set* of adherent points associated with the *proposition* $P$, even though $P$ is an equivalence class of sets. Hence we may unambiguously speak of $adh(P)$ even when P is a proposition and not just a set of points.

The following topological fact will be useful when we compare a proposition $P$ to its double negation $\neg\neg P$.

**Theorem 5.** For any proposition $P$, $adh(\neg\neg P) \subseteq adh(P)$.

For a proposition $P$ in bivalent open phase space logic, we saw that a measurement might verify $P$ or it might not verify $P$. However, if a measurement does not verify $P$ we cannot conclude that this measurement gives us that $P$ is false. We will now describe a three-valued open phase space logic in which propositions *may* be naturally assigned values of "true" or "false" by a given measurement. The price to be paid is that a given measurement may also result in a value of "indeterminate" being assigned to $P$. We shall assign truth values according to the following criteria:

**Definition 6:** Let $P$ be a proposition in open phase space logic and let $m$ be a measurement result. Suppose that $P_i$ is any set in $[P]$. Then the proposition $P$ will be assigned the truth value
1) **true** if $m$ verifies $P$;
2) **false** if $m$ does not verify $P$ but $m \cap adh(P_i) = \emptyset$;
or
3) **indeterminate** if $m$ does not verify $P$ but $m \cap adh(P_i) \neq \emptyset$.

A few observations concerning this definition are in order. In view of our measurement theory, these three truth values are sufficient for all possible measurements of a system designed to verify a proposition $P$. Theorem 4 implies that all sets in a given proposition (an equivalence class) will have the same set of adherent points. Hence the assignment provided by Definition 6 is well-defined. As measurements are identified with open sets, any measurement of $P$ which contains adherent points of a set $P_i \in P$ must also contain interior points of $P_i$. Thus there exists a more precise measurement consistent with the current measurement which might verify $P$; i.e., the measurements which contain adherent points of the representatives of $P$ are precisely those measurement which do not verify $P$ and were inconclusive as to whether or not $P$ is false.

We now turn our attention to statements whose interpretations in bivalent open phase space logic differ from the interpretations in three-valued open phase space logics. Again, complete proofs of these facts can be found in [3]. Recall that in bivalent open phase space logic our " truth values " were "verified " and "unverified". One of the motivations for the three-valued logic presented here is to recover a natural notion for the valuation of propositions as being “true” or “false”. Of course the notions of verified and truth are related: We say, using the three-valued logic, that a measurement determines that a proposition is true if the measurement verifies the proposition and a proposition is determined to be false if the measurement verifies the negation of the proposition. As we have seen, we cannot equate the ideas of “ true ” and “verified” as it is conceivable that neither a proposition nor its negation will be verified by a suite of measurements. This is the reason why we need the value of “indeterminate” along with those of “true” and “false”.

Theorem 7 is an example of a statement which can be made using three-valued open phase space logic which has no natural analogue in the bivalent open phase space logic. We may think of this theorem as a weaker form of *tertium non datur*.

**Theorem 7.** The proposition $P \vee \neg P$ is never false (i.e. no measurement can verify the negation of $P \vee \neg P$).

The question arises: What can be said about $\neg P$ and $\neg\neg P$ given that a measurement assigns a particular truth value to $P$? Theorems 8 to 10 and Corollary 11 provide the answer.

**Theorem 8.** If a measurement assigns true to $P$, then the same measurement assigns false to $\neg P$ and true to $\neg\neg P$.

**Theorem 9.** If a measurement assigns false to $P$, then the same measurement assigns true to $\neg P$ and false to $\neg\neg P$.

**Theorem 10.** If a measurement assigns indeterminate to $P$, then the same measurement may assign true or indeterminate to $\neg\neg P$.

**Corollary 11.** If a measurement assigns indeterminate to $P$, then the same measurement may assign false or indeterminate to $\neg P$.

## V. Three-valued Open Phase Space Logic and Billiard Ball Computation

The three-valued open pase space logic was used to analize several physical systems in [3]. We now turn to the analysis of billiard ball computation using the three-valued interpretation of open phase space logic. Let us again consider the billiard ball interaction gate shown in Figure 1.

The theory of measurement is unchanged from that described in Section III above. Similarly, the considerations of resolution and tolerance are largely unchanged. The difference between the analyses is in the values that measurements assign to propositions. For example, we will say the gate assigns the proposition " A and B" a value of *true* if ball A is located in interval $y_1$ and ball B is located in interval $y_4$. Recall that the bivalent open phase space logic assigned a value of "verified" to the proposition "A and B" under these circumstances.

Similarly, the gate will return a value of "B and not A is true" if and only if $m_B \subset (y_2 - \epsilon, y_2 + \epsilon)$ and $m_A \not\subset Y$ where $Y$ is the entire right hand wall of the gate. The gate will return a value of "A and not B is true" if and only if $m_A \subset (y_2 - \epsilon, y_2 + \epsilon)$ and $m_B \not\subset Y$.

Recall that, in the bivalent variant, it is possible that a measurement $m$ will assign no value to a given proposition (see the discussion at the end of Section III.). Let us reconsider the example involving the interaction gate where the measurement $m$ with resolution $\delta$ is the interval $(y_1 + \epsilon - \delta/2, y_1 + \epsilon + \delta/2)$ (see figure 3.). In the three-valued logic, the proposition "$A$ and $B$ " is assigned a value of "indeterminate". In the three-valued open phase space logic no measurement fails to assign one of the three values to a give proposition. Thus, this three-valued logic provides a more natural analysis of billiard ball computers than does the bivalent open phase space logic.

This three-valued logic is not entirely satisfactory. The results IV.8 to IV.11 illustrate an important fact about the operators in three-valued open phase space logic: the operators are not truth functional. That is to say, a given truth value for a measurement - proposition pair $(m, P)$ does not give a well-defined truth value for an operator acting on $P$ under $m$. Corollary IV.11 provides a prime example of this lack of truth functional behavior: If $P$ is assigned the value indeterminate by $m$, then $\neg P$ may be assigned indeterminate or false by $m$.

This lack of truth functionality is shared by other derived logics: the bivalent open and closed phase space logics, and the von Neumann - Birkhoff logic for quantum mechanics are all examples of non-truth functional logics. For a discussion of this property for the von Neumann - Birkhoff logic see [11]. That the bivalent phase space logics are not truth functional is closely related to the fact that the three-valued variety of open phase space logic is not truth functional. As we noted above, it is possible for a proposition and its negation to both be unverified. Of course it is also possible that a proposition may be unverified and for its negation to be verified. Thus the negation operator in open phase space logic under the bivalent interpretation is not truth functional.

The fact that these logics are not truth functional is closely related to a perhaps more obvious property which they share: they are not truth-valued. That is, the connectives in these logics do not assign truth values to propositions, rather they assign other propositions to propositions. This contrasts with the situation found in classical logic where connectives assign truth values to sets of propositions. Indeed, the situation in classical logic is that we may think of operators as being either truth-valued or proposition valued. Assigning a proposition to an operation acting on other propositions is equivalent to assigning a truth value to the operation based only on the truth values of the propositions upon which the operation is acting. With the derived logics listed above these two assignments are no longer equivalent.

In [3] a modified phase space logic was described in which negation is truth functional, at least upon a restricted class of propositions (equivalence classes of sets). As this restriction is somewhat *ad hoc* and as there exists another three valued derived logic which is truth functional we will not consider the modified three valued phase space logic here. Instead, we will now describe this other three valued logic. This logic is referred to as the "twin open set phase space logic" for reasons which will soon be made clear.

## VI. Properties of Twin Open Set Phase Space Logic

The twin open set phase space logic is suggested by the observation that there are two ways a measurement assigns a truth value to a proposition concerning a physical system. One of these ways is to use the measurement to *verify* the proposition. We say that a measurement $m$ verifies $P$ if $m \subset P_0$, where $P_0$ is the canonical representative of $P$. The other way to use the measurement is to *falsify* the proposition. We say that a measurement $m$ falsifies $P$ if $m \subset (\overline{P_0})^c$, where $(\overline{P_0})^c$ is the complement of the closure of the canonical representative of $P$. Note that the set $(\overline{P_0})^c$ is an open set. These dual notions were used in [2] to describe the open phase space logic and its dual, the closed phase space logic. In twin open set phase space logic, these two notions are used simultaneously to assign truth values to a proposition.

In order to consider simultaneously the verifiability and falsifiability of a proposition we will not consider single equivalence classes of subsets of the phase space; rather, we consider ordered pairs of certain equivalence classes of subsets of the phase space. That is, in twin open set phase space logic, we define propositions as follows:

**Definition 1:** We say that $P$ is a proposition in twin open set phase space logic if $P = ([V_0], [F_0])$, where $V_0$ and $F_0$ are disjoint open sets. Here, $[X]$ denotes the equivalence class of all sets that have the same interior as the set $X$.

Given this notion of proposition, we now define the assignment of truth values as follows:

**Definition 2:** Let $P = ([V_0], [F_0])$ be a proposition in twin open set phase space logic and let $m$ be a measurement. The proposition $P$ will be assigned the truth value
1) **true** if $m \subset V_0$;
2) **false** if $m \subset F_0$;
or
3) **indeterminate** otherwise.

We will refer to $[V_0]$ as the "verifiability class" of $P$ and $[F_0]$ as the "falsifiability class" of $P$. We will refer to $V_0$ as the "verifiability set of $P$" and to $F_0$ as the "falsifiability set of $P$".

Note that the definitions of "true" and "false" are not identical to the notions of "verifiability" and "falsifiability" discussed above. If such were the case then propositions would always be of the form $P = ([V_0], [(\overline{V_0})^c])$. That is, the falsifiability set would always be the interior of the complement of the closure of the verifiability set. Thus, propositions would be entirely characterized by their verifiability sets (or by their falsifiability sets) alone. The collection of propositions in the three-valued logic developed here is much larger; the only requirement for the canonical twin open sets is that they must be disjoint. This broader collection of propositions reflects that an experimenter's resources for verifying a statement are often not identical to those resources that falsify the same statement.

Note also that a subset $S$ is always an element of the verifiability class of some proposition ($S$ is also the element of the falsifiability class of the negation of $P$; a fact that will be explained in the following). We need only let $V_0 = int(S)$ and then let $F_0$ be some open set disjoint with $int(S)$. There will always be such an $F_0$; even in spaces with minimal separation properties (e.g. non-Hausdorff) we can at least take $F_0 = \emptyset$. Thus, there will be no restriction to "proper propositions" as in the modified three-valued phase space logic.

We now define logical operators for any two propositions $P$ and $Q$. We will take

$$P = ([P_v], [P_f]) \tag{30}$$
$$Q = ([Q_v], [Q_f]) \tag{31}$$

where $A_v$ is any representative of the verifiability class of the proposition $A$ and $A_f$ is any representative of the falsifiability class of the proposition $A$.

**Definition 3.** The logical operators $\neg$ (negation), $\vee$ (disjunction), and $\wedge$ (conjunction) are defined as follows:

$$P \vee Q = ([int(P_v) \cup int(Q_v)], [int(P_f) \cap int(Q_f)]), \tag{32}$$

$$P \wedge Q = ([int(P_v) \cap int(Q_v)], [int(P_f) \cup int(Q_f)]), \tag{33}$$
$$\neg P = ([int(P_f)], [int(P_v)]). \tag{34}$$

We cast these definitions in terms of the open sets $int(P_v), int(P_f)$, etc. because each equivalence class contains exactly one open set. This set can be taken as the canonical representative for the equivalence class. In each definition we apply set operations on the canonical representative verifiability and falsifiability sets for each proposition. The definitions are thus unambiguous.

The interpretation of these operations in terms of measurements is straightforward. For example, a measurement $m$ will assign a value of "true" to the proposition $P \wedge Q$ if and only if $m$ assigns a value of "true" to both propositions $P$ and $Q$. That is, $m$ assigns a value of "true" to $P \wedge Q$ if and only if $m \subset int(P_v) \cap int(Q_v)$. Alternatively, $m$ will assign a value of "false" to the proposition $P \wedge Q$ if and only if $m$ assigns a value of "false" to either proposition $P$ or $Q$. That is, $m$ assigns a value of "false" to $P \wedge Q$ if and only if $m \subset int(P_f) \cup int(Q_f)$. $m$ will assign a value of "indeterminate" to the proposition $P \wedge Q$ if and only if $m \not\subset int(P_v) \cup int(Q_v)$ and $m \not\subset int(P_f) \cap int(Q_f)$.

In previous discussions of phase space logics [2,3], no attempt was made to define an implication operation. Indeed, as was pointed out in [3], the only derived logic for which a natural implication was known to exist was the logic referred to as the "logic derived from the unphysical theory of measurement." (The "unphysical theory of measurement" is the one which allows for measurements of infinite precision. In this logic the lattice of propositions is the Boolean lattice of subsets of the phase space.) In contrast with the other logics derived from more realistic theories measurement, an implication operation *will* be defined for the twin open set phase space logic. The definition is as follows:

**Definition 4.** The logical operator $\rightarrow$ (implication) is defined as follows:

$$P \rightarrow Q = \neg P \vee Q \tag{35}$$
$$= ([int(P_f) \cup int(Q_v)], [int(P_v) \cap int(Q_f)]) \tag{36}$$

This operation also has a natural interpretation: a measurement $m$ will assign a value of "true" to the proposition $P \rightarrow Q$ if and only if $m$ assigns a value of "true" to either $\neg P$ or $Q$. That is, $m$ assigns a value of "true" to $P \rightarrow Q$ if and only if $m \subset int(P_f) \cup int(Q_v)$. Alternatively, $m$ will assign a value of "false" to the proposition $P \rightarrow Q$ if and only if $m$ assigns a value of "false" to both $\neg P$ and $Q$. That is, $m$ assigns a value of "false" to $P \rightarrow Q$ if and only if $m \subset int(P_v) \cap int(Q_f)$. $m$ will assign a value of "indeterminate" to the proposition $P \rightarrow Q$ if and only if $m \not\subset int(P_f) \cup int(Q_v)$ and $m \not\subset int(P_v) \cap int(Q_f)$.

We will now state some results about the operations in the twin open set phase space logic.

**Theorem 5.** The operations $\vee$ and $\wedge$ are commutative and associative. $\vee$ distributes over $\wedge$ and vice versa.

**Proof.** In order to exhibit the character of the derivations, we will give only the proof of the distributivity of $\wedge$ over $\vee$. Other parts of the proof are equally straightforward.

Suppose A, B, and C are propositions in the twin open set phase space logic. Then

$$A \wedge (B \vee C) = ([A_v], [A_f]) \wedge ([int(B_v) \cup int(C_v)], [int(B_f) \cap int(C_f)]) \tag{37}$$
$$= ([int(A_v) \cap int\{int(B_v) \cup int(C_v)\}], [int(A_f) \cup int\{int(B_f) \cap int(C_f)\}]) \tag{38}$$

The union of open sets is open, so $int(int(V) \cup int(W)) = (int(V) \cup int(W)$. Also, the interior of an intersection of sets is equal to the intersection of their interiors, so $int(int(V) \cap int(W)) = (int(V) \cap int(W)$. Thus

$$A \wedge (B \vee C) = ([(int(A_v) \cap int(B_v)) \cup (int(A_v) \cap int(C_v))], \tag{39}$$
$$[(int(A_f) \cup int(B_f)) \cap (int(A_f) \cup int(C_f))]) \tag{40}$$
$$= ([(int(A_v) \cap int(B_v))], [(int(A_f) \cup int(B_f))]) \tag{41}$$
$$\vee ([(int(A_v) \cap int(C_v))], [(int(A_f) \cup int(C_f))]) \tag{42}$$
$$= (A \wedge B) \vee (A \wedge C). \tag{43}$$

Therefore $\wedge$ distributes over $\vee$ in the twin open set phase space logic. The proof that $\vee$ distributes over $\wedge$ is the dual of the proof given: simply replace $\vee$ with $\wedge$,$\cap$ with $\cup$ and vice versa. □

The next several theorems highlight the main difference between the twin open set phase space logic and the phase space logics that have been described previously: negation in the twin open set phase space logic has many of the properties of the negation in standard Boolean logic. In particular, negation is an involutive operation; i.e., $\neg\neg P = P$ for all propositions $P$. Also, versions of the Law of Noncontradiction and *tertium non datur* hold in twin open set phase space logic.

**Theorem 6.** In twin open set phase space logic, $\neg(\neg P) = P$ for all propositions $P$.

**Proof.** Let $P = ([P_v], [P_f])$ be a proposition in twin open set logic. By the definition of negation we have

$$
\begin{aligned}
\neg(\neg P) &= \neg([P_f], [P_v]) && (44)\\
&= ([P_v], [P_f]) && (45)\\
&= P. \square && (46)
\end{aligned}
$$

The following theorem can be thought of as a version of the Law of Noncontradiction. In standard Boolean logic $P \wedge \neg P$ is always false. In the twin open set logic this proposition is never true; i.e. for different measurements and propositions it may receive a value of indeterminate or a value of false.

**Theorem 7.** In the twin open set phase space logic, for any proposition $P$, $P \wedge \neg P = ([\emptyset], [U])$. That is, $P \wedge \neg P$ is a proposition which is never assigned the value of true ($U$ is some open subset of the topological space $X$).

**Proof.** Let $P$ be any proposition in the twin open set phase space logic. We then have

$$
\begin{aligned}
P \wedge \neg P &= ([P_v], [P_f]) \wedge ([P_f], [P_v]) && (47)\\
&= ([int(P_v) \cap int(P_f)], [int(P_f) \cup int(P_v)]) && (48)\\
&= ([\emptyset], [U]), && (49)
\end{aligned}
$$

where $U = int(P_f) \cup int(P_v)$, an open subset of $X$. $\square$

By a dual argument, we also have a version of the Law of the Excluded Middle (*tertium non datur*). As a middle truth value is allowed, we might wish for a more appropriate name.

**Theorem 8.** In the twin open set phase space logic, for any proposition $P$, $P \vee \neg P = ([U], [\emptyset])$. That is, $P \vee \neg P$ is a never false proposition (again, $U$ is some open subset of the topological space $X$.

**Proof.** Let $P$ be any proposition in the twin open set phase space logic. We then have

$$
\begin{aligned}
P \vee \neg P &= ([P_v], [P_f]) \vee ([P_f], [P_v]) && (50)\\
&= ([int(P_v) \cup int(P_f)], [int(P_f) \cap int(P_v)]) && (51)\\
&= ([\emptyset], [U]), && (52)
\end{aligned}
$$

where $U = int(P_f) \cup int(P_v)$, an open subset of $X$. $\square$

De Morgan's laws also hold in twin open set phase space logic.

**Theorem 9.** For propositions $P$ and $Q$ in twin open set phase space logic,

$$\neg(P \wedge Q) = \neg P \vee \neg Q, \tag{53}$$
$$\neg(P \vee Q) = \neg P \wedge \neg Q. \tag{54}$$

**Proof.** Let $P = ([P_v], [P_f])$ and $Q = ([Q_v], [Q_f])$ be propositions in twin open set phase space logic. We then have

$$\begin{aligned}
\neg(P \wedge Q) &= \neg([int(P_v) \cup int(Q_v)], [int(P_f) \cap int(Q_f)]) && (55)\\
&= ([int(P_f) \cap int(Q_f)], [int(P_v) \cup int(Q_v)]) && (56)\\
&= ([int(P_f)], [int(P_v)]) \vee ([int(Q_f)], [int(Q_v)]) && (57)\\
&= ([P_f], [P_v]) \vee ([Q_f], [Q_v]) && (58)\\
&= \neg P \vee \neg Q. && (59)
\end{aligned}$$

By a dual argument, we see that De Morgan's other law holds as well. □

As Theorems 7 and 8 illustrate, there are many propositions that are never false and there are many propositions that are never true. Among such propositions, there are two worthy of special mention: The proposition $1 = ([X], [\emptyset])$ is the always true, never false proposition. Dually, the proposition $0 = ([\emptyset], [X])$ is the always false, never true proposition. In the following theorem, we see that 0 and 1 act as identity elements for $\vee$ and $\wedge$ respectively.

**Theorem 10.** For each proposition $P$ in the twin open set phase space logic we have

$$P \vee 0 = P \tag{60}$$
$$P \wedge 1 = P \tag{61}$$

**Proof.** Let $P = ([P_v], [P_f])$ be any proposition, we have

$$\begin{aligned}
P \vee 0 &= ([P_v], [P_f]) \vee ([\emptyset], [X]) && (62)\\
&= ([P_v \cup \emptyset], [P_f \cap X]) && (63)\\
&= ([P_v], [P_f]) && (64)\\
&= P. && (65)
\end{aligned}$$

The proof that $P \wedge 1 = P$ is equally straightforward. □

We see by the theorems presented above that the propositions in the twin open set logic, under the operations of $\vee$ and $\wedge$, satisfy many of the properties of a Boolean algebra. Recall [17] that an algebra $< L, \oplus, \otimes, \tilde{}, 0, 1 >$ is said to be *Boolean* if it satisfies the following axioms:

(i) $\oplus$ and $\otimes$ are associative binary operators;
(ii) $\oplus$ and $\otimes$ are symmetric;
(iii) 0 and 1 are the identities of $\oplus$ and $\otimes$ respectively;
(iv) the unary operator $\tilde{}$ satisfies $b \oplus \tilde{b} = 1$ and $b \otimes \tilde{b} = 0$;
(v) $\oplus$ distributes over $\otimes$ and vice-versa.

We see that (iv) is the only axiom for a Boolean algebra that twin open set phase space logic fails to satisfy. As this logic satisfies weakened forms of the Law of Non-contradiction and *tertium non datur*, it may fail to be Boolean. It should be noted that if $X$ is given the trivial (or indiscrete) topology; then the derived twin open set phase space logic is Boolean. It is also interesting to note that the twin open set phase space logic derived from such a topology is effectively bivalent, in that no measurement will ever assign a value of indeterminate to a proposition.

One motivation for developing the modified phase space logic was to exhibit a derived logic where the negation operator is truth functional [3]. We recall when an operation is said to be truth functional: Given a fixed measurement $m$ and propositions $P$ and $Q$, we say that the operation $X$ on $P$ and $Q$ is *truth functional* if the truth values of $P$ and $Q$ uniquely imply the truth value of $X(P, Q)$ under the measurement $m$. It is easy to see that the negation operator is truth functional; however, the disjunction operator is not truth functional.

This lack of truth functionality can be seen in the following counterexample. Let the phase space be the Euclidean space $\mathbb{R}$; let $P$ be the proposition $P = ([(-2, 2)], [\emptyset])$ and let $Q$ be the proposition $Q = ([(1, 5)], [\emptyset])$. The measurement $m = (0, 6)$ assigns "indeterminate" to $P$, $Q$ and to $P \vee Q$. In contrast, the measurement $m = (0, 3)$ assigns "indeterminate" to $P$ and to $Q$ but it assigns "true" to $P \vee Q$.

This example points out that truth functionality is a fairly strong requirement in that the operations must assign truth values uniquely for *all* measurements $m$. Indeed, one might object that the second measurement considered in the example is too big. The measurements $m = (0, 2)$ or $m = (1, 3)$ as being more meaningful for determining the value of $P \vee Q$. In fact, if $m$ is any measurement contained in (1,3) we see that the disjunction operation is truth functional for such an $m$.

These considerations lead us to define the notion of *weakly truth functional* as follows:

**Definition 11.** Given propositions $P$ and $Q$, we say that the operation $X$ on $P$ and $Q$ is *weakly truth functional* if there exists a measurement $m$ such that the truth values of $P$ and $Q$ uniquely imply the truth value of $X(P, Q)$ under $m$.

If an operation is made weakly truth functional by a measurement $m$ it is also weakly truth functional under any measurement $n$ such that $n \subset m$. For this reason we say that the notion of weak truth functionality is robust with regard to refinements of measurements. It is straightforward to show that $\vee$ and $\wedge$ in twin open set phase space logic are weakly truth functional.

## VII. Twin Open Set Phase Space Logic and Billiard Ball Computation

The analysis of billiard ball computation using twin open set phase space logic is, in many respects, the same as the analysis done using the modified three-valued logic described in Section V. As was mentioned in the discussion of the modified three-valued logic, there is a restriction to proper propositions. Roughly speaking, a proposition is said to be *proper* if the canonical representative contains no "gray regions". More precisely, $P$ fails to be proper if there is an open subset $U$ of $X$ such that $P_0 \cap U$ is dense in $U$ with empty interior.

The analysis using the twin open set logic does differ from the earlier analysis in at least one feature: the identity of the falsifiability set for a particular proposition. Let us again consider the interaction gate displayed in Figure 1. In the three-valued open phase space logic the proposition "A and B" is assigned a value of false if and only if $m_A \in int[Y \setminus (y_1+\epsilon, y_1-\epsilon)]$. "A and B" is assigned a value of indeterminate if and only if $m_A \cap adh(y_1+\epsilon, y_1-\epsilon) \neq \emptyset$. This make sense mathematically but one is left to wonder what this means in terms of the physical set up of the gate. How could a propositon ever be assigned a value of "indeterminate" by a real live interaction gate?

Now consider the gate using the twin open set logic. Setting the verifiability set as $(y_1+\epsilon, y_1-\epsilon)$ does not automatically provide the identity of the falsifiabilty set. We could make the same identification as was done in the three-valued open phase space logic, but we can chose many other possibilties. For example, we could make $(y_2+\epsilon, y_2-\epsilon) \cup (y_3+\epsilon, y_3-\epsilon)$ the falsifiability set for "A and B". This make a certain amount of sense; each set in the union is the verifiability set for a "competing" proposition. One can imagine setting up "chutes" for the billiard balls at the endpoints of each of our intervals as shown in Figure 4. In such a case we know that "A and B" will be assigned a value of indeterminate if ball A is found outside any of the chutes centered at $y_1$, $y_2$ or $y_3$. Thus, using the twin open set logic allows us to physically implement the assignment of the indeterminate truth value.

The twin open set phase space logic also has another, mathematically, desirable feature: there is no restriction to proper propositions. A proposition is determined once a pair of disjoint open sets is specified. Given a set $S$, there are several propositions that can be associated with $S$. One such proposition was alluded to at the beginning of the previous section: for the set $S$ we can consider the proposition $P = ([int(S)], [(\overline{int(S)})^c])$ (as $(\overline{int(S)})^c$ is open and disjoint from $int(S)$ it is a proposition in twin open set phase space logic).

## VIII. Summary and Future Work

The design of circuitry in current computers is based on Boolean algebra theory, and reasoning about this circuitry is carried out using Boolean propositional logic, a natural result of the fact that both Boolean circuitry and propositional logic with appropriate definitions of operations, duals, and identities form Boolean algebras. The models of computation, on which existing computers are based, assume that physical measurement can be made precise. A fact that is not only false, but that acts as a limiting factor when considering how computation can be carried out.

Here, we have considered nonstandard logics to analyze one model of computation, the billiard ball model. We chose this model because of its conservative nature, and we chose nonstandard logics for the analysis, because this analysis allows us to accept the physical reality that measurement is not precise. Using both two valued and three valued derived logics we examined what can be said about the billiard ball model. In particular, we noted that the twin open set phase space logic satisfies all but one of the axioms for Boolean algebras, a fact that enables many of the common properties such as De Morgan's laws to hold.

Having completed our analysis of the billiard ball interaction gate, we are now considering composition of gates in this model. Since most computations consist of composing multiple gates, we need to continue our application of nonstandard logics to more complicated computations. Once compositions are introduced, we must deal with the problem of error propagation, i.e., as computation progresses through a model, error builds up exponentially, thereby making results meaningless after even a small number of computations. In the conservative billiard ball model we hope to find a way to carry out error correction at each step during the computation process by using the "extra" information available in the output of each gate. That is, as conservative gates provide one to one mappings from input to output, we will consider how to use this for error correction instead of energy conservation.

Another question we plan to address concerns the mathematical system one gets when using a derived logic. In particular, we have already shown for the twin open set phase space logic that we have a mathematical system that is almost a Boolean algebra. We want to see what theorems result from this system and what, if any, natural circuitry can be designed.


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## References


[1] G. Birkhoff and J. von Neumann, *Ann. of Math.* **37**, 823–43 (1936).

[2] M. D. Westmoreland and B. W. Schumacher, *Phys. Rev. A* **48**, 977–85 (1993).

[3] M. D. Westmoreland, B. W. Schumacher, and S. Bailey *International Journal of Theoretical Physics*, **35**, 31.

[4] Steven Vickers, *Topology Via Logic*, Cambridge Tracts in Theoretical Computer Science 5 (Cambridge University Press, Cambridge, 1989).

[5] J. Łukasiewcz, *Selected Works*, L. Borkowski, ed. (North-Holland Publishing Company, Amsterdam, 1970); cited in [16].

[6] H. Reichenbach, *Philosophic Foundations of Quantum Mechanics* (University of California Press, Berkeley and Los Angeles, 1944, 1965); sections 29-37 reprinted in *The Logico-Algebraic Approach to Quantum Mechanics* C. A. Hooker, ed. (Reidel, Dordrecht-Holland, 1975).

[7] J. B. Rosser, *American Journal of Physics* **9**, 207-212 (1941); cited in [9].

[8] C. I. Lewis and C. H. Langford, *Symbolic Logic* (The Century Company, New York, 1932);cited in [9]

[9] J. B. Rosser and A. R. Turquette, *Many-valued Logics* (North-Holland Publishing Company, Amsterdam, 1952).

[10] J. F. Munkres, *Topology: A First Course* (Prentice-Hall, Englewood Cliffs, NJ, 1974).

[11] P. Gibbins, *Particles and Paradoxes: The Limits of Quantum Logic* (Cambridge University Press, Cambridge and New York, 1987).

[12] C. G. Hempel, *The Journal of Symbolic Logic* **10**, 97-100 (1945); cited in [16].

[13] A. R. Turquette, *The Philosophic Review* **54**, 513-516 (1945); cited [16].

[14] E. Nagel, "Logic without Ontology," in *Naturalism and the Human Spirit*, Y. H. Krikorian, ed. (Columbia University Press, New York, 1944), 210-241; reprinted in *Readings in Philosophical Analysis*, H. Feigl and W. S. Sellars, eds. (Appleton, New York, 1949), 191-210; cited in [16].

[15] P. T. Geach, *Logic Matters* (University of California Press, Berkeley and Los Angeles, 1972).

[16] M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1974).

[17] D. Gries and F. B. Schneider, *A Logical Approach to Discrete Math* (Springer-Verlag, New York, Berlin, and Heidelberg, 1994)