ZENO'S ARROW AND CLASSICAL PHASE SPACE LOGICS

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Received August 11, 1993; revised January 24, 1994

We analyze the Zeno's familiar paradox of the arrow using recently developed non-Boolean derived logics for classical systems. We show that the paradox depends upon a premise that is identically false in such logics, so that the language of experimental propositions is immune to the paradox.

Key words: logic, Zeno's paradoxes, measurement.

1. ZENO'S ARROW

In a famous 1936 paper, von Neumann and Birkhoff [1] suggested that some of the paradoxes of quantum mechanics could be addressed by applying a nonstandard logic to propositions concerning quantum mechanical systems. For example, the counter-intuitive results of the familiar double-slit experiment illustrate the non-distributive character of the quantum logic of von Neumann and Birkhoff. In [2] we have shown that other theories besides quantum mechanics naturally lead to derived logics of experimental propositions when an explicit account of measurement is included. In particular, we have shown that a simple measurement theory for classical mechanics gives rise to *nonstandard* derived logics. These classical phase space logics differ from quantum logic in that it is negation that is nonstandard instead of distributivity.

Our purpose here is to show that nonstandard derived logics for classical mechanics can be applied to classical paradoxes in a way analogous to the application of quantum logic to the counter-intuitive features of quantum theory. In both situations, the connections of experimental observation to the underlying theory is non-trivial. The paradoxical aspects of each theory arise in part from the subtle nature of the link between observation and theory. This is reflected in the non-Boolean properties of the derived logics.

Although one does not usually dwell on this in an introductory physics course, there are apparent paradoxes related to classical mechanics. The most famous of these are the paradoxes of motion put forward by Zeno of Elea in about 500 B.C. These paradoxes supported the Eleatic contention that the universe was an undivided and unchanging unity and that the analysis of space and time into separate points and moments omitted essential aspects of that unity [3]. For example, the paradox of the arrow runs:

> ...Zeno argues that an arrow in flight is always at rest. At any given instant, he claims, the arrow is where it is, occupying a portion of space equal to itself. During the instant it cannot move, for that would require the instant to have parts, and an instant is by definition a minimal and indivisible element of time. If the arrow did move during the instant it would have to be in one place at one time of the instant, and in in a different place at another part of the instant. [4]

If the arrow is at rest at any given instant, how can the arrow move?

Many resolutions to this paradox have been proffered. One of the more widely accepted resolutions (cf. Salmon [4] and Grünbaum [5]) of the paradox of the arrow is the one proposed by Bertrand Russell, which has come to be known as the "at-at" theory of motion. Wesley Salmon gives the following description ([4], p. 41):

> Let us now apply [the] conception of a mathematical function to the motion of an arrow; to keep the arithmetic simple let it travel at the uniform speed of ten feet per second in a straight line, starting from $x = 0$ at $t = 0$. At any subsequent time t, its position $x = 10t$. ...The requirement is that the arrow be at the appropriate point at the appropriate time - nothing is said about the instantaneous velocity of the arrow as it occupies each of these

> points. . . . the motion itself is described by the pairing of positions with times alone.

As an account of what one means when one says "the motion of the arrow is described by a function" the "at-at" theory is certainly correct. However,if one is concerned about how one arrives at such a function, there is a problem. When a physicist states that a function describes the flight of the arrow what is meant experimentally is that a series of measurements has been performed and the given function fits the observed data. If one is mindful of a physical theory of measurement there is no way to arrive at a unique function as the proper descriptor of the arrow's flight.

In [2] we argue that in any physically motivated theory of measurement for classical systems the following assumptions should be reflected:

1. Observables are continuous phase space functions.
2. Values of observables cannot be determined with infinitely high precision.
3. Only finitely many measurement outcomes are available at one time.

The problem for the "at-at" theory is that no set of measurements satisfying these three postulates suffice to specify a *unique* function which describes the motion of the arrow. Certainly, given a finite set of time and position measurements of the arrow, it is possible to find a function consistent with them. But so will an uncountable infinity of other functions. Indeed, it can be shown that, given any set of measurements which satisfy the above postulates, one can find an uncountable number of functions with almost any "nice" mathematical properties (such as continuity, differentiability, or even analyticity) which will provide an experimentally reasonable "at-at" description of the arrow's flight. Conversely, given a function which provides an appropriate "at-at" description, no set of measurements which satisfy the postulates will *verify* the given function to the exclusion of all others. Thus, even though a function may satisfy the requirements of the "at-at" theory, that function can not be uniquely provided or verified by measurement. We may therefore say that the "at-at" theory does not operate at the level of physical measurement.

One is then left to wonder about the status of the paradox of the arrow at the level of measurement. If we are left with Zeno's paradoxical situation unresolved at the level of measurement, this might be sufficient reason to avoid discussions of physically motivated theories of measurement! Fortunately, this turns out not to be the case. In [2] we discuss "derived logics" where at least some of the propositions are sanctioned by results of measurements (this is what is meant by our phrase "at the level of physical measurement"). We will show that the premise for the paradox of the arrow is in fact identically false in these logics. Before we demonstrate this, we will first provide a brief description of these derived logics for classical systems.

2. CLASSICAL PHASE SPACE LOGICS

One way to think about the implications of a measurement theory is to discuss the *derived logics* of experimental propositions that result from the theory. For classical systems, we have previously developed "closed" and "open" phase space logics derived from a rudimentary theory of classical measurements [2]. These logics, also called "falsifiability" and "verifiability" logics, can be used to examine the basis for the paradox of the arrow. In the following sections we will review some of the properties of these logics, use them to analyze aspects of the paradox of the arrow, and discuss some of the implications.

The measurement theory for the two classical phase space logics that we develop is based upon the three postulates given above. Postulate (1) expresses the fact that a classical measurement process is itself a continuous dynamical evolution of a system of interest and a measurement device. Postulate (2) implies that the value of an observable is only localized *within some open interval* by a measurement. A measurement therefore only localizes the state of the system of interest to an open set in phase space. Postulate (3) simply indicates that the limitations imposed by Postulates (1) and (2) cannot be evaded by limiting processes involving an infinite number of measurements.

Thus classical measurements, in their most generic form, provide the answers to questions of the form "Is the state of the system within the open set $\mathcal{O}$?" Two propositions about the state of the system are experimentally equivalent if they cannot be distinguished from one another by the answers to any finite set of such questions. They are effectively the same proposition. There are at least two possible senses in which two propositions are "indistinguishable" and thus equivalent:

1. The assertions would be falsified by exactly the same measurement results.
2. The assertions would be verified by exactly the same measurement results.

A proposition in one of our derived logics is actually an equivalence class of indistinguishable phase space propositions according to one of the two senses above.

A phase space proposition can be identified with a subset of the phase space X—namely, the set of points in phase space for which the proposition is true. Therefore, a proposition in our derived logics can be identified with an equivalence class of phase space sets that are indistinguishable by a finite number of open sets (which represent measurements). As before, there are two distinct meanings of "indistinguishable":

1. Two sets are indistinguishable if they intersect exactly the same open sets. This means that the two corresponding phase space propositions are consistent with exactly the same measurement results, so this definition agrees with the "falsification" one above. Two sets that are indistinguishable in this sense have the same closure.
2. Two sets are indistinguishable if they contain exactly the same open sets. This means that the two corresponding phase space propositions are implied by exactly the same measurement results, so that this definition agrees with the "verification" one above. Two sets that are indistinguishable in this sense have the same interior.

We call the derived logic resulting from criterion (1) the closed phase space logic (or falsifiability logic), and the logic resulting from criterion (2) the open phase space logic (or verifiability logic).

We will describe the closed phase space logic and apply it to analyze the paradox of the arrow. The corresponding discussion for the open phase space logic will be provided in the Appendix.

The fact that propositions of our logics are identified with *equivalence classes* of sets rather than the sets themselves requires us to take care in defining the logical operators of negation ($\neg$), conjunction ($\wedge$), and disjunction ($\vee$). In the Boolean logic associated with raw subsets of the phase space, these operations are defined in terms of ordinary set operations: complement, intersection, and union. In the derived logics, this approach must be modified to meet two requirements: first, the result of any operation must itself be an equivalence class of sets; and second, each operation must be well-defined, i.e., must be independent of the equivalence class representatives that are used for its definition.

In the closed phase space logic, this is done by defining operations in terms of the closures of the representative sets. For example, suppose A and B are propositions in the closed phase space logic, corresponding to equivalence classes $[A_\mu]$ and $[B_\nu]$ of subsets of the phase space. The sets in each equivalence class have a common closure: $\overline{A_\mu} = \overline{A_\nu}$ for all sets A_μ and A_ν in the equivalence class A (the overline "$\overline{X}$" symbol reprepresents set closure).We can define our logical operators as follows (the superscript "c" represents set complementation):

$$\neg A \;=\; [(\overline{A_\mu})^c],$$

$$\begin{aligned} A \vee B &= [\overline{A_\mu} \cup \overline{B_\nu}], \\ A \wedge B &= [\overline{A_\mu} \cap \overline{B_\nu}]. \end{aligned}$$

Since all sets in a given equivalence class have the same closure, each equivalence class contains exactly one closed set. This set can be taken as the canonical representative for the equivalence class. In each definition, we apply the usual set operations on the canonical representative set for each proposition. The definitions are thus unambiguous.

Two propositions in this logic are especially important: that represented by the whole space X and that represented by the null set. The equivalence class for the entire space is the "never falsifiable" statement, so it is given a special name: $[X] = 1$. Similarly, the equivalence class of the null set is the "always falsifiable" statement, so we denote it by $[\phi] = 0$.

Many of the properties of classical logic also hold for the closed phase space logic: $\vee$ distributes over $\wedge$ and vice versa. However, the nonstandard nature of negation induces several non-Boolean features to this logic: the "Law of Noncontradiction" ($A \wedge \neg A = 0$) does not hold in the closed phase space logic. On the other hand, *tertium non datur* the "law of the excluded middle" ($A \vee \neg A = 1$) does hold.[1] (According to a duality principle, the situation in the open phase space logic is exactly reversed, with the law of noncontradiction holding but not the law of the excluded middle.)

An example is in order. Suppose a particle is constrained to move in one dimension, like a train on its track. At any moment, the location and velocity of the train are only known to a finite precision. Consider the following statement:

> At time $t = t_0$, the train is between 1240.1 and 1240.2 meters from the beginning of the track, and is traveling between 21.2 and 21.3 meters per second.

(We will ignore the fact that *time* measurements can only be made with finite precision. For our present purposes, this is an unnecessary complication.)

This statement corresponds to a proposition in the closed phase space logic, the equivalence class A of subsets of the phase space X containing the closed set

$$A_\mu = \{(x, v) \in X | 1240.1 \leq x \leq 1240.2, 21.2 \leq v \leq 21.3\}.$$

We should point out that the equivalence class A contains a great many other sets, such as

$$A_\nu = \{(x, v) \in A_\mu | x, v \text{ are both rational}\}.$$

A_μ and A_ν are equivalent because no measurement can distinguish between the points in A_ν (which have rational phase space coordinates) and the points in A_μ.

In our analysis of the paradox of the arrow we will need the following properties of closed phase space logic:

$$\neg(\neg A) \sqsubseteq A \quad , \quad \neg 1 = 0 \ ,$$
$$\neg 0 = 1 \quad , \quad A \wedge 0 = 0 \ .$$

(The symbol $\sqsubseteq$ represents containment of the canonical representative sets of the propositions.) A more complete discussion of these properties can be found in Sec. II of [2].

3. ANALYSIS OF THE ARROW PARADOX

We begin our analysis of the arrow paradox by focusing on one of Zeno's premises. As Salmon puts it, "At any given instant ... the arrow is where it is, occupying a portion of space equal to itself." Thus, the *tip* of the arrow occupies a single point. Considering an arrow moving in one dimension, we can say this as:

> At any moment $t = t_0$, the position x of the tip of the arrow is precisely $x = x_0$.

To say that the tip is *precisely* at $x = x_0$ is in fact to assert that the position takes this value and no other—that is, we really have

> At any moment $t = t_0$, the position x of the tip of the arrow is $x = x_0$. Furthermore, for any $y_0 \neq x_0$, $x \neq y_0$ at this moment.

Zeno depends on this more elaborate premise to argue that the arrow is at rest during the moment.

That is, if we set

P = At the moment $t = t_0$, the position of the tip of the arrow is precisely at $x = x_0$.

Q = At the moment $t = t_0$, the position of the tip of the arrow is at some other position $x \neq x_0$.

Using this notation we see that the reformulation of Zeno's premise is of the form $P \wedge \neg Q$.

We now show that this stronger statement is false at the object language level (that is, in the language of physical propositions that are sanctioned by physical measurements) when the object language logic is the closed (falsifiability) phase space logic. In terms of the mathematical structure of the logic, we see that

$$\begin{aligned} Q &= [\{(x,p) \in X | x \neq x_0\}] \\ &= [\overline{\{(x,p) \in X | x \neq x_0\}}] \\ &= [X] \; = \; 1. \end{aligned}$$

Even though negation is not standard in closed phase space logic, we do have $\neg 1 = 0$; therefore, we may conclude that

$$P \wedge \neg Q = P \wedge 0 \; = \; 0.$$

Hence this premise of the arrow paradox is false in the closed phase space logic.

One might wonder why we chose to label the second conjunct as $\neg Q$ instead of $\neg(\neg P)$, which it certainly is. By using the new label we are emphasizing the difference between $\neg(\neg P)$ and P. Recall that, in general, in the closed phase space logic we have that $\neg(\neg P) \sqsubset P$. Thus, even though we can say that $\neg Q = \neg(\neg P)$ is false—that is, equivalent to the always falsifiable statement 0—we can not conclude that the statement represented by P is false.

Considering this situation in physical terms, we recall that the outcome of a measurement localizes the system to some open set in phase space. Any such set that includes points with $x = x_0$ must necessarily also include distinct points (an infinite number, in fact) nearby. Thus, at the level of experiment, $\neg P$ is never falsifiable, so $\neg P = 1$. Hence $\neg Q = \neg(\neg P)$ is equivalent to the "always false" statement in the object language that uses the closed phase space logic.

The following objection might be raised at this point: If the path of the tip of the arrow can be described by some well-defined mathematical function (in the strict mathematical sense) then the tip must be located at some particular position at some particular time. But how do we know that a particular mathematical function describes the path of the tip of the arrow? The only things which sanction our claims of truth or falsehood for certain propositions are the results of measurements having limited precision. It may well be true (in fact, if the theoretical set-up is a good one it must be true) that a particular mathematical function describes the path of the tip of the arrow to within experimental error; but to assert that what the arrow is really doing is occupying particular positions at particular times is to assert something beyond the ability of physical measurements to determine.

We need not deny the existence of moments and exact positions. However, if moments and exact positions do exist, we have no way of asserting this at the object level, i.e., the level of experimental propositions. In spite of these cautionary remarks, we do note that the analysis of Zeno's premise based on the closed phase space logic is somewhat reminiscent of Aristotle's approach to the paradox. Aristotle denied that the paradox had any meaning in as much as moments do not exist. As we have made the simplifying assumption that time can be precisely determined, our analysis says of spatial positions what Aristotle has to say of moments; we could make the analogy closer by dropping our simplifying assumption. His argument is as follows:

> The third argument, the one just stated, is that the moving arrow is standing. This follows from assuming that time is composed from nows; for if one does not grant this, there will be no syllogism. [7]

This remark reminds us that there is a substantial difference between a *point* in $\mathbb{R}$ and an *interval*. This difference is worth keeping in mind. The velocity at any moment of time is defined as a derivative; but this derivative is defined as the limit of average velocities over smaller and smaller intervals of time. In other words, to define instantaneous velocity we need to have the trajectory of the particle already defined for an entire interval of time (albeit perhaps a small one).

This raises interesting questions about causality. In classical physics, the laws of motion are ordinary differential equations with time as the independent variable. In Hamiltonian mechanics, for example, we have first-order equations of motion for position and momentum variables (q_i and p_i, respectively):

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \ ,$$

where $H(p_i, q_i)$ is the Hamiltonian function. Initial values of q_i and p_i determine a unique solution to these equations, so in a sense we can say that the future trajectory of the system is determined by its position and momentum at a single moment. The initial conditions "cause" the future behavior of the system via the laws of motion. However, the definition of the time derivatives on the left-hand side of the equations of motion requires that the trajectory *already be defined for an interval of time that contains moments from both the future and the past of the moment in question.* Equations of motion only make sense if the trajectory is "already exists" over an entire interval of time.

Our approach to the paradox does bear some resemblance to the "Aristotelian" resolution: there is an essential difference between *arbitrarily* small scales and *infinitely* small scales. On the other hand, while Aristotle claims that isolated moments do not exist, we make no such strong claim. Our rudimentary measurement theory merely implies that measurement does not sanction statements about points and moments. We are therefore led to consider the logic of empirical statements about physical systems. As was pointed out in [2], the logic of physical systems is not necessarily the standard Boolean logic.

Another way of putting this is that there are unobservable aspects of mathematical models for classical mechanics. These unobservable aspects reside in the relation of points to intervals—i.e., in the *infinitesimal* structure of $\mathbb{R}$. This can be seen by noting the development of nonstandard analysis. Ordinary classical mechanics can be recast using a nonstandard version of $\mathbb{R}$ that includes infinitesimal quantities. By the transfer principle [8], we know that the calculated results of such a theory would be identical to those of the standard theory, so that the two theories would be observationally equivalent. They differ only in the infinitesimal structure of the underlying mathematical model.

Once we confine ourselves to the results of measurements, which cannot explore this infinitesimal structure, we will be led to use derived logics, which need not be standard. They differ from ordinary logic not only in the calculus of logical connectives but also in the status of truth claims within the logic. In standard logic, the truth value of a given statement is *a priori* relative to the logic. That is, the claim that something is true is sanctioned by considerations outside the logic. This contrasts with the situation concerning physical systems. The truth or falsity of a statement about a physical system is sanctioned by measurement, and as measurement is itself a physical process this truth or falsity is not *a priori* relative to the logic.

In the closed phase space logic, the non-Boolean nature of the negation leads us to say that Zeno's hypothesis about the arrow is false at the level of the object language. In the language of the standard mathematical model, the statement concerning Zeno's arrow is perfectly true, and leads us to Russell's "at-at" theory of motion. As we have pointed out, however, the basic "at-at" statements of this theory cannot be experimentally sanctioned.[2]

The trajectory of the arrow is a mathematical idealization that cannot be fully justified on empirical grounds. Any finite set of measurements of finite precision can only provide an *approximate* trajectory for the arrow, and no approximate trajectory (however precisely determined) is exact. Thus, the notion of an exact trajectory is problematic *even in classical mechanics*. In quantum theory, of course, exact trajectories do not even exist, and the notion of a trajectory breaks down on small scales (determined by the size of $\hbar$).

Indeed, Richard Schlegel [9] has given a quantum mechanical resolution to Zeno's paradox of the arrow. Schlegel's resolution depends on the uncertainty relations, in that precise information about the arrow's velocity excludes precise information about the arrow's position, and vice-versa. Schlegel, like the present work, builds his resolution by considering *what can be measured.* We have argued that one does not need the machinery of quantum mechanics to produce a resolution along these lines; classical mechanics, if viewed though the proper logical lenses, can also provide a resolution to Zeno's arrow paradox.

4. APPENDIX: THE ARROW IN OPEN PHASE SPACE LOGIC

In the open (verifiability) phase space logic, propositions are associated with equivalence classes of subsets of the space, but now two subsets are equivalent if and only if their interiors are equal. In this logic, a proposition A is verifiable only if there is some measurement of finite precision which can localize the system to an open set in phase space entirely contained in every set in A. This means that any statement of the form "the position is $x = x_0$" is unverifiable and therefore false (equivalent to 0) in an object language which uses the open phase space logic. This is due to the fact that the subset of phase space determined by the property $x = x_0$ has an empty interior.

The operations in the open phase space logic are defined in terms of the interiors of the representative sets. All sets in a given equivalence class have the same interior, so this is the only open set in that equivalence class. If we denote the equivalence class of A_μ in the open phase space logic by

$$A^\circ = [A_\mu]_\circ = [\mathrm{int}(A_\mu)]_\circ,$$

the logical operators are defined as follows:

$$\begin{aligned} \overset{\circ}{\neg}\, A^\circ &= [(\mathrm{int}(A_\mu))^c]_\circ, \\ A^\circ \overset{\circ}{\vee} B^\circ &= [\mathrm{int}(A_\mu) \cup \mathrm{int}(B_\nu)]_\circ, \\ A^\circ \overset{\circ}{\wedge} B^\circ &= [\mathrm{int}(A_\mu) \cap \mathrm{int}(B_\nu)]_\circ. \end{aligned}$$

In a way that is also analogous to the closed phase space logic, the "always verifiable" proposition $[X]_\circ$ is denoted by 1 and the "never verifiable" proposition $[\phi]_\circ$ by 0. The closed and open phase space logics are related by a "duality theorem" (cf. Sec. II of [2]).

One of the conjuncts of the revised statement of the arrow paradox asserts that the tip of the arrow is at one particular point at one particular time. This statement defines a "thin" subset of the phase space, a subset with empty interior, so that it is in the equivalence class $[\phi]$, the "never verifiable" proposition. This time, we can state Zeno's premise in the form

$$P^\circ \overset{\circ}{\wedge} \overset{\circ}{\neg}\, Q^\circ$$

in the object language. In this case we see that

$$Q^\circ = [\{(x,p) \in X | x \neq x_0\}]_\circ,$$

so

$$\begin{aligned} \overset{\circ}{\neg}\, Q^\circ &= [(\mathrm{int}(\{(x,p) \in X | x \neq x_0\})^c)]_\circ \\ &= [\{(x,p) \in X | x = x_0\}]_\circ \\ &= [\phi]_\circ \;=\; 0. \end{aligned}$$

Note that in the open phase space logic we have $P^\circ \overset{\circ}{\subseteq} \overset{\circ}{\neg} (\overset{\circ}{\neg}\, P^\circ)$ as well as $0 = \overset{\circ}{\neg} (\overset{\circ}{\neg}\, 1)$. Thus, in the open phase space logic version, both conjuncts of the premise are assigned the always false (i.e., never verifiable) truth value.

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NOTES

1. It may seem odd to the reader that we claim here that negation is nonstandard but that *tertium non datur* does hold in closed phase space logic. That the former is true is proved by exhibition of an example. Let us suppose that our topological space is $\mathbb{R}^2$ and let $A = [\{(0,y)|y \in \mathbb{R}\}]$; i.e., A is the class of all subsets of $\mathbb{R}^2$ whose closure is the y axis. This implies that

$$\begin{aligned} \neg A &= [\{(x,y)|x \neq 0\}] = [\overline{\{(x,y)|x \neq 0\}}] \\ &= [X] = 1. \end{aligned}$$

   But, as we note in the text, $\neg 1 = 0$ and so $\neg(\neg A) = \neg 1 = 0$ giving us an example of $\neg(\neg A) \neq A$. That *tertium non datur* holds is proved as follows. By the definitions of $\vee$ and $\neg$ we have

$$A \vee \neg A = [\overline{A_\mu}] \vee [\overline{(\overline{A_\nu})^c}],$$

   where A_μ and A_ν are arbitrary representatives of A. Let us choose both to be the canonical closed representative A_Γ. Then

$$A \vee \neg A = [A_\Gamma \cup \overline{(A_\Gamma)^c}].$$

   Since $V \cup V^c = X$ for any subset V of X, and since $V \subseteq \overline{V}$, it follows that $A_\Gamma \cup \overline{(A_\Gamma)^c} = X$. Thus, $A \vee \neg A = 1$, and *tertium non datur* holds.
2. These considerations lead us to wonder whether dynamical laws might be re-phrased in terms that better reflect the experimental facts of life. Instead of merely constructing derived logics, we might also calculate using a mathematics based on functions with experimentally meaningful domains and ranges—for example, we might describe a "generalized trajectory" as a function from *regions* of time to *regions* of phase space. Such an analysis is beyond the scope of the present paper.