Correction to Three-valued derived logics for classical phase spaces

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Definition 1: For a set P_i in the topological space X, we define the *proposition* corresponding to P_i (denoted by P) to be the equivalence class $[P_i]$ defined by the equivalence relation of having the interior *and* the same boundary as P_i. I.e., $P_j \sim P_i$ (and so $P_j \in [P_i] = P$) if and only if $\operatorname{int} P_j = \operatorname{int} P_i$ and $\partial P_j = \partial P_i$.

The verification that the condition "same interior and same boundary as" gives an equivalence relation is straightforward. Unfortunately, this definition does not allow for definitions of connectives in terms of set operations as is the case for other derived logics for classical systems which have been previously described. This situation is remedied by the following two definitions and result; these provide a useful characterization of almost all (this restriction will be made clear by Definition 3.) of the equivalence classes:

Definition 2: Let S be a set in the topological space X. Then a point y in the boundary of S is a *recreant point* if and only if y is not an adherent point of S and every neighborhood of y contains a point of the interior of S^c. The set of all the recreant points of S is denoted by $rctS$.

Definition 3: Let S be a set in the topological space X. Then a point y in the boundary of S is a *penumbral point* if and only if y is not an adherent point of S and there exists a neighborhood U of y such that $\operatorname{int}(ScapU) = \emptyset$ and $\operatorname{int}(S^c capU) = \emptyset$. The set of all penumbral points of S is denoted by $pnmb(S)$.

Theorem 1. The boundary of a set S is the disjoint union of the adherent, recreant, and penumbral points of S; i.e.,

$$\partial S = adh(S) \cup rct(S) \cup pnmb(S)$$

and $adh(S) \cap rct(S) = rct(S) \cap pnmb(S) = adh(S) \cap pnmb(S) = \emptyset$.

Proof. By the definitions, the sets $adh(S)$ and $rct(S)$ are disjoint. If $x \in adh(S)$ then each neighborhood of x has a nonempty intersection with $\operatorname{int} S$. So, for every neighborhood U of x we have

$$\begin{aligned} \operatorname{int} S \cap U &= \operatorname{int} S \cap \operatorname{int} U = \\ \operatorname{int} S \cap U &\neq \emptyset, \end{aligned}$$

thus $x \notin pnmb(S)$ hence $adh(S) \cap pnmb(S) = \emptyset$. Similarly, if $x \in rct(S)$ then every neighborhood of x has a nonempty intersection with $\operatorname{int} S^c$. So, for every neighborhood U of x we have

$$\operatorname{int} S^c \cap U = \operatorname{int} S^c \cap \operatorname{int} U =$$

$$\text{int}\, S^c \cap U \quad \neq \quad \emptyset,$$

thus $x \notin pnmb(S)$ hence $rct(S) \cap pnmb(S) = \emptyset$.

That $adh(S) \cup rct(S) \cup pnmb(S) \subseteq \partial S$ follows directly from the definitions; thus, we need only to prove the reverse containment. Assume that $x \in \partial S$ but $x \notin adh(S)$. Thus, there exists a neighborhood of U of x such that $U \cap \text{int}\, S = emptyset$. As $x \in \partial S$, every neighborhood of x has a nonempty intersection with S^c. Either, every neighborhood of x must intersect $\text{int}\, S^c$ or there is a neighborhood V of x such that $V \cap \text{int}\, S^c = \emptyset$. If the former condition holds, $x \in rct(S)$. If the latter condition holds then $W = (U \cap V)$ is a neighborhood of x such that $W \cap \text{int}\, S = \emptyset$ and $W \cap \text{int}\, S^C = \emptyset$. Thus, $x \in pnmb(S)$. Thus, $\partial S \subseteq adh(S) \cup rct(S) \cup pnmb(S)$ and equality now follows. □

Theorem 2. For sets S_i, S_j, such that $pnmb(S_i) = \emptyset = pnmb(S_j)$ the condition $[\text{int}\,(S_i) \cup rct(S_i)] = [\text{int}\,(S_j) \cup rct(S_j)]$ is equivalent to the condition that $\text{int}\,(S_i) = \text{int}\,(S_j)$ and $\partial S_i = \partial S_j$.

Proof. Suppose that $[\text{int}\,(S_i) \cup rct(S_i)] = [\text{int}\,(S_j) \cup rct(S_j)]$ and let $x \in \text{int}\,(S_i)$. We wish to show that $x \in \text{int}\,(S_j)$. As the interior of a set and its boundary are disjoint sets, it is the case that either $x \in \text{int}\,(S_j)$ or $x \in rct(S_j)$, but not both. Assume that $x \in rct(S_j)$, the contradiction we obtain proves that $x \in \text{int}\,(S_j)$.

By definition, there exist neighborhoods N_{xi} of x such that $N_{xi} \subseteq \text{int}\, S_i$ and N_{xj} of x such that $N_{xj} \cap \text{int}\, S_j^c \neq \emptyset$. As $x \in N_{xi} \cap N_{xj}$ we know that $N_{xi} \cap N_{xj} \neq \emptyset$. Also, since $N_{xi} \cap N_{xj}$ is a neighborhood of x and x is a recreant point of S_j we know that $(N_{xi} \cap N_{xj}) \cap \text{int}\, S_j^c \neq \emptyset$. As every set appearing in $(N_{xi} \cap N_{xj}) \cap \text{int}\, S_j^c$ is open, we have that the set

$U = (N_{xi} \cap N_{xj}) \cap \text{int}\, S_j^c$ is an open set. We note that $U \subseteq \text{int}\,(S_i)$ and that $U \subseteq \text{int}\,(S_j^c)$. Thus, every point of U is in $\text{int}\,(S_i)$ and so is not in $rct(S_i)$. Also, every point of U is in $\text{int}\,(S_j^c)$ and so is not in $\text{int}\,(S_j)$ and is not in $\partial(S_j)$, in particular, no point of U in $rct(S_j)$. Thus, there is some point $u \in U$ such that $u \in (\text{int}\,(S_i) \cup rct(S_i))$ and $u \notin (\text{int}\,(S_j) \cup rct(S_j))$. This contradiction of our assumption proves that $x \in \text{int}\,(S_j)$. As x is an arbitrary element of $\text{int}\,(S_i)$, we have shown that $\text{int}\,(S_i) \subseteq \text{int}\,(S_j)$. A symmetric argument proves that $\text{int}\,(S_j) \subseteq \text{int}\,(S_i)$. Thus we have that $\text{int}\,(S_i) = \text{int}\,(S_j)$.

As the interior of a set and its boundary are disjoint this result also yields the fact that $[\text{int}\,(S_i) \cup rct(S_i)] = [\text{int}\,(S_j) \cup rct(S_j)]$ implies that $rct(S_i) = rct(S_j)$. Our result together with Theorem III.5 of [**?**] also implies that $adh(S_i) = adh(S_j)$. That $\partial S_i = \partial S_j$ now follows by Theorem 1 and the condition that $pnmb(S_i) = \emptyset = pnmb(S_j)$.

We now prove the converse: $\text{int}\,(S_i) = \text{int}\,(S_j)$ and $\partial S_i = \partial S_j$ implies that $[\text{int}\,(S_i) \cup rct(S_i)] = [\text{int}\,(S_j) \cup rct(S_j)]$. As $pnmb(S_i) = \emptyset = pnmb(S_j)$ and $\partial S_i = \partial S_j$ we only need to show that $adh(S_i) = adh(S_j)$ since Theorem 1. then implies that $rct(S_i) = rct(S_j)$ and so the result will follow. Assume that there is a point x such that $x \in adh(S_i)$ but $x \notin adh(S_j)$. As $x \notin S_j$ we conclude that there is a neighborhood U of x such that $U \cap \text{int}\, S_j = \emptyset$. As $\text{int}\,(S_i) = \text{int}\,(S_j)$ we then conclude that $U \cap \text{int}\, S_i = \emptyset$ so $x \notin adh(S_i)$ which contradicts the assumption about x. Thus $adh(S_i) \subseteq adh(S_j)$; a symmetric argument shows that $adh(S_j) \subseteq adh(S_i)$ and so $adh(S_i) = adh(S_j)$ □

The condition placed on the penumbral points in Theorem 2. motivates the following definition:

Definition 4: A proposition P is said to be *proper* if and only if $pnmb(P_i) =$

$\emptyset$ for every element P_i of P).