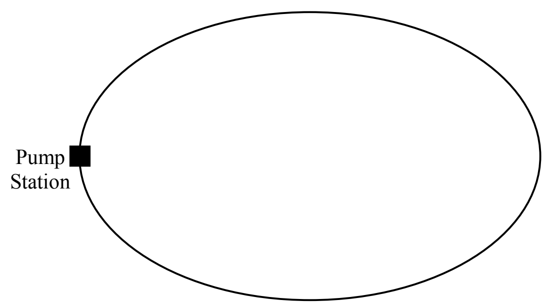
The Mean Value Theorem

Abstract:
This lab provides an introduction to the Mean Value Theorem. Section 1 develops intuition through Rolle's Theorem. In Section 2, we generalize Rolle's Theorem resulting in the Mean Value Theorem.

Introduction:
An oil company has decided to upgrade one of its pipelines and return it to full operational capacity. The pipeline consists of five pipe sections connecting six Pump Stations. Among the upgrades required is the installation of flow meters to carefully monitor the flow rates. In addition to minimal and maximal flow rates, which are typically measured at the Pump Stations, the company would also like to keep records on the average flow rate between Pump Stations.

It has been suggested that the easiest method of recording the average flow rate is to locate electronic flow meters at the locations where the average flow rate is actualized. To do this we must mathematically represent the fluid flow through each section of the pipeline and determine the locations of the average flow rates. Thus, our problem reduces to locating a position for which the instantaneous rate of change is equal to the average rate of change.

Section 1: Closed Loop Fluid Flow
Our problem has been reduced to locating a position where the instantaneous rate of change is equal to the average rate of change. Let's consider a simplified problem of fluid flow in a closed circular loop. We will measure as our variable the horizontal distance away from the Pump Station.



Open the Maple Worksheet: **Mean Value Theorem** and work through Exploration 1 Section 1 for a visualization of the fluid flow through the closed loop.

Exercise 1.1

a) What is the average flow rate through the closed loop? Would this be the average flow rate through any closed loop, regardless of shape? Explain.

b) Determine the position where the instantaneous flow rate is equal to the average flow rate. Is there a physical significance to this location? Explain.

c) For such a closed loop flow problem, would you always be guaranteed a position for which the instantaneous flow rate is equal to average flow rate. If so, is this position necessarily unique? Explain.

Mathematical Question:

The closed loop problem above can be generalized mathematically with the following question:

Given a function $f(x)$ *on the interval* $[a,b]$ *such that* $f(a) = f(b)$*, is there a location the interval* (a,b) *on where the slope of the tangent line equals the slope of the secant line?*

If $f(a) = f(b)$ then the line containing the points $(a, f(a))$ and $(b, f(b))$ must be a horizontal line with slope $m = 0$. Also, since the slope of the tangent line is given by the derivative we can simplify our question as:

Given a function $f(x)$ *on the interval* $[a,b]$ *such that* $f(a) = f(b)$*, is there a point c on the interval* (a,b) *where* $f'(c) = 0$*?*

Before attempting to answer the question, lets consider a few graphical examples.

Exercise 1.2:

a) For each of the following, use Maple to plot the given function on the interval $[a,b]$. Does there appear to be locations on the interval with horizontal tangent lines. Determine these locations by solving the equation $f'(c)=0$.

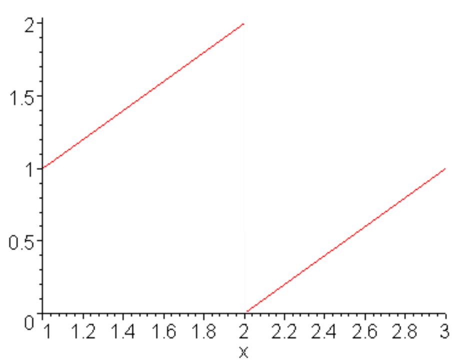
$f(x)=x^2-5x+4$ on the interval $[1,4]$.

$f(x)=x^3-x^2-2x+1$ on the interval $[-1,2]$.

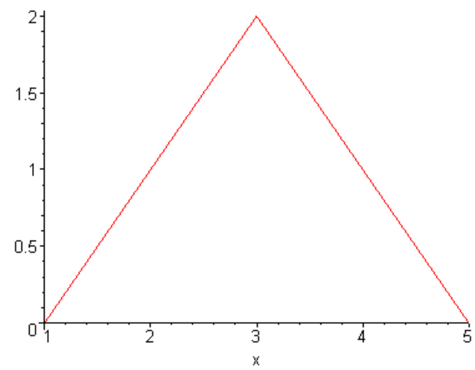
$f(x)=4$ on the interval $[-1,1]$.

b) How does the *extreme value theorem* (Theorem A p162) help to answer our mathematical question?

c) The graph of $f(x)=\begin{cases} x, & x<2 \\ x-2 & x\geq 2 \end{cases}$ is given on the interval $[1,3]$. Is there a position with a horizontal tangent line? Explain how this example confirms or contradicts your intuitive answer to (Part b) and why.



d) The graph of $f(x) = -|x-3|+2$ is given on the interval $[1,5]$. $[1,3]$. Is there a position with a horizontal tangent line? Explain how this example confirms or contradicts your intuitive answer to (Part b) and why.



Rolle's Theorem

(Part c) and (Part d) of Example 1.2 illustrate the two conditions that form exceptions to the answer of our mathematical question. The answer is to the question is known as **Rolle's Theorem** and is stated as follow

Let f *be a function satisfying the following hypotheses:*

1) f *is continuous on the closed interval* $[a,b]$.

2) f *is differentiable on the closed interval* (a,b).

If $f(a) = f(b)$ *then there is a number c in the open interval* (a,b) *such that* $f'(c) = 0$.

Open the Maple Worksheet: **Mean Value Theorem** and work through Exploration 1 Section 2 for intuition on why this theorem holds when the hypothesis is met.

Section 2: The Mean Value Theorem

Mathematical Question

We were originally interested in the flow of oil through a pipeline between pumping stations. Suppose the fluid is pumped from Station A at time $t = a$ and arrives at Station B at time $t = b$. If we use the function $f(t)$ to describe the position of the fluid at time t, then the average flow rate of the fluid traveling from Station A to Station B is given by

$$\frac{f(b) - f(a)}{b - a}.$$

In that we are looking for the position, provided it exists, along the pipeline where the instantaneous flow rate is equal to the average flow rate, our mathematical question is given by the following:

Given a function $f(x)$ *on the interval* $[a,b]$, *is there a location the interval* $[a,b]$ *on where the slope of the tangent line equals* $\frac{f(b)-f(a)}{b-a}$ *?*

We saw how Rolle's Theorem provided an answer to the mathematical question relating to the closed loop flow problem. But how, if at all, does it apply to our pipeline fluid flow problem. We can immediately see that Rolle's theorem does not directly apply because the purpose of our pipeline to move the oil from one location to another. Thus the starting point is different from the terminal point. Further we are not interested in finding locations where the flow rates are zero, rather we are interested in finding the locations of average flow rates.

But there is some similarity, note that the zero flow rate is the average flow rate for a closed loop. Thus it may be possible to manipulate Rolle's Theorem to obtain a more general result.

Return to the Maple Worksheet: **Mean Value Theorem** and work through Exploration 2.

Exercise 2.1:

a) Using the results of Maple Exploration 2 for intuition, explain in your own words how Rolle's Theorem relates to solve our current mathematical question.

We saw in the Maple Exploration 2 that our new problem is just a tilted version of our old problem. In other words if we can tilt a continuous and differentiable function enough (by exactly the opposite amount of tilt of the secant line) so as to get a new function for which Rolle's Theorem applies then we will be able to find tangent lines with slope equal to the slope of the secant line. This results in the following **Mean Value Theorem**

Let f *be a function satisfying the following hypotheses:*

1) f *is continuous on the closed interval* $[a,b]$.
2) f *is differentiable on the closed interval* (a,b).

then there is a number c in the open interval (a,b) *such that* $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Math 123 **Name: [Click here and type name]**

The Mean Value Theorem

Problem Set:

1) Give an explanation of why continuity and differentiability would be required for the Mean Value Theorem.
[Click **here** and type response]

2) Open the Maple Worksheet: **Mean Value Theorem** Exploration 3. Here you will find the function describing the flow of oil between Pump Stations.

   a) If we wanted to use the Mean Value Theorem to determine the locations of average flow rates between pumping stations, what intervals would you use for each section of pipe.
   [Click **here** and type response]

   b) Used graphical methods to approximate the tangent lines at the locations of average slope between Station 1 and Station 2 and between Station 2 and Station 3. Give the equation of those tangent lines.
   [Click **here** and type response]

   c) Determine analytically, all possible locations for the flow meters between Station 1 and Station 2 and between Station 2 and Station 3.
   [Click **here** and type response]