A Brief LaTeX Example*

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We have examined a few topologies on the real line. Next, let's look at the plane, $\mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, two points in $\mathbb{R}^2$, we previously introduced the Euclidean distance formula

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

For each x in $\mathbb{R}^2$, define $B(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$. Each $B(x, \epsilon)$ is the open ball of radius ϵ centered at x. Define

$$\mathcal{B} = \{B(x, \epsilon) | x \in \mathbb{R}^2, \epsilon > 0\};$$

that is the collection of open balls associated with d.

Theorem 1. *$\mathcal{B} = \{B(x, \epsilon) | x \in \mathbb{R}^2, \epsilon > 0\}$ is a basis for $\mathbb{R}^2$.*

Before proceeding with the proof of the theorem, we introduce a lemma whose proof we ask you to provide later.

Lemma 1. *Assume $x \in B(y, r)$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset B(y, r)$.*

The lemma indicates that if a point x is in some open ball B, then there is an open ball centered at x contained in B as well (Figure 1). We use the lemma in establishing that $\mathcal{B}$ satisfies the second property of a basis.

Proof of the Theorem. Since each point $x \in \mathbb{R}^2$ is contained in the basis element $B(x, 1)$, the first condition of being a basis is satisfied. Now we need to check that if x is a point in the intersection of two basis elements, there is a basis element containing x and contained in the intersection. Let $x \in B(y, r_1) \cap B(z, r_2)$. By the lemma there exist $\epsilon_1, \epsilon_2 > 0$ such that $B(x, \epsilon_1) \subset B(y, r_1)$ and $B(x, \epsilon_2) \subset B(x, r_2)$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $B(x, \epsilon) \subset B(x, \epsilon_1) \cap B(x, \epsilon_2) \subset B(y, r_1) \cap B(z, r_2)$, completing the proof that $\mathcal{B}$ is a basis. □

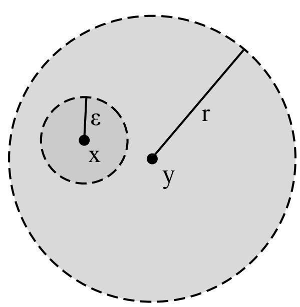

Figure 1: Every point x in $B(y, r)$ is the center of some ball contained in $B(y, r)$.

*Taken from pp. 9–10 of *Some Topology Book* by Someone.