Generalizations of the Tonnetz: Tonality Revisited

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ABSTRACT

In this paper, we attempt to explore mathematical structures of tonal music from the 18th and 19th centuries. We review the known mathematical/musical structures and, noting lacking features of this geometrical construct as a useful model of tonal harmony, we propose generalizations that may be better suited to tonal music. We bring several mathematical devices to bear on the Tonnetz (or *Tone Network*) in new ways, including Cayley graphs and Coxeter hyperbolic representations. We conclude with three-dimensional geometric models that represent the four-note seventh chords which are ubiquitous in 18th and 19th century music.

1. INTRODUCTION

The intellectual endeavor to codify and clarify music in mathematical terms dates at least to the ancient Greeks and the Pythagoreans. Composers of the Medieval and Renaissance periods developed techniques that a modern mathematician would recognize by principles of symmetry and transformation. As late as the 17th and 18th centuries, great scientific minds such as Newton and Euler turned their attention to music theory as well. However during the 18th century, music theory began to develop into an academic area independent of its scientific roots. With the work of Rameau [1], a theory of tonal harmony gained strength and held sway until the end of the 19th century. The music of this era — known to musicologists as the *common practice period* or the *tonal period* — was based upon a theoretical underpinning that was largely independent of explicit mathematical influence.

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With the birth of post-tonal and atonal music at the close of the 19th century, the established harmonic order failed. Theorists turned again to mathematics for organizing principles around which to build a new theoretical framework. At the turn of the 20th century, one influential direction (presaged in the work of Leonard Euler two centuries prior [10]) was that of Otto Riemann [1], whose two-dimensional geometric representation of fundamental harmonic structures, the *Tonnetz*, (German for "Tone Network"), gave rise to a rich Neo-Riemannian theory of set-theoretic transformations in pitch-class space. This has proven to be extremely fertile ground for music analysis. The group theoretical work begun by Lewin [3] has many contemporary scholars.

In the current work, we look for applications of Riemann's Tonnetz to the *tonal* music of the common practice period. We review the known mathematical structure of the Tonnetz and, noting awkward features of this geometrical model, propose generalizations and modifications that may be better suited to tonal music. We bring several mathematical devices to bear on the Tonnetz in novel ways, including Cayley graphs and Coxeter hyperbolic representations. We conclude with three-dimensional geometric models that represent four-note seventh chords which are vital in compositions of the 18th and 19th centuries.

1.1 Algebra

While mathematical constructs have been used to study music since ancient times, the use of group theory is a relatively new and powerful approach to this topic [3]. We gather a few key definitions that will be used below [5][7].

Definition 1. *Let G be a set together with a binary operation that assigns to each ordered pair (a, b) of elements of G an element in G denoted by ab. We say G is a* group *under this operation if the following properties are satisfied:*

1. *Associativity. The operation is associative; that is, $(ab)c = a(bc)$ for all $a, b, c \in G$.*
2. *Identity. There is an element e in G such that $ae = ea = a$ for all $a \in G$. We call e the* identity *element.*
3. *Inverses. For each element A in G, there is an element b in G such that $ab = ba = e$.*

Definition 2. *If a subset H of a group G is itself a group under the operation of G, we say that H is a* subgroup *of G. A group G is* cyclic *if it is of the form $\langle a\rangle = \{a^n | n \in Z\}$ and we say that a is a* generator *of the group.*

We will call the cyclic subgroup $\langle a\rangle$ in the group G the *cycle* generated by a. We will explore relations between tonality and dihedral groups.

Definition 3. *A* dihedral group *of order $2n$ is the group generated by two elements: a and b under the three following conditions:*

- $a^n = e$,
- $b^2 = e$, *and*
- $abab = e$,

where e is the identity element. We refer to the dihedral group of order $2n$ as D_n.

1.2 Music Theory

Pitch is an auditory phenomenon in which the brain analyzes the frequency of sound heard and assigns it a musical tone. For our purposes, the psycho-perceptual aspects of pitch will be neglected — we will consider a pitch to be characterized by its frequency. In Western music, we assign pitches a letter, with or without an accidental (sharp or flat), and an octave. For our purposes, we will use *pitch classes*, which ignore octave differences between pitches, and an *equal tempered scale*, which considers, for instance, D♭ and C♯ as the same pitch. The result is a collection of twelve pitch classes, labeled alphabetically A to G and including A♭, B♭, D♭, E♭ and G♭. This collection is simply one octave of the piano keyboard. From a mathematical standpoint, we assign numerical values to the pitch classes and work with modular arithmetic modulo 12. By convention, we assign C:$= 0$, D♭ $:= 1, \ldots,$ B:$= 11$. We refer to the *interval* between two pitch classes as the distance between two pitch classes, that is, the minimum number of semi-tones between pitches.

A *chord* is a collection pitch classes, and we assign the names *trichord* for a three-note chord and a *tetrachord* for a four-note chord. We will indicate a collection of pitch classes by $[x_1, x_2, ...]$ where each x_i represents the numerical value assigned to that pitch class.

2. THE TONNETZ

2.1 Major and minor triads

Major and minor triads are the basis of music theory, so studying how they are related can illuminate more complicated musical structures. A *triad* is a trichord where the intervals between adjacent pitches comprise either three or four semitones — the minor third or major third intervals, respectively. In the major triad, the pitches are arranged so that we have a minor third stacked above a major third. For example, the C-major triad contains the pitch classes C, E and G. Any permutation or multiplicity of these notes is considered a C-major triad, (like $[7, 0, 7, 4]$ or $[4, 7, 0]$), but out of convenience, we will consider CEG ($[0, 4, 7]$) the canonical C-major triad. A minor triad is the stacking of a major third above a minor third. So the C-minor triad is CE♭G ($[0, 3, 7]$). In each case, we regard C (or 0) as the *root* of the triad. By fiat, we refer to an M-major triad as M and an M-minor triad as m, where M and m represent the pitch class name of the root of the chord.

2.2 Chords on the Tone Clock

Approaching triads as stacking of thirds clearly shows a relation between the major and minor triads. They are related by a flipping, permutation or “inversion” of the intervals. We will explore the effects of flipping intervals using a polygonal representations around a *tone clock*. The tone clock is a representation of the 12 pitch classes of Western music arranged chromatically around a circle, analogous to the hours of an analog clock [8]. We can represent a chord of any size (up to twelve notes) as a single, simple (no edges intersect), convex (all interior angles are less than 180 degrees) polygon by connecting every pitch in the chord with exactly two edges. We do so in such a way so that an n-chord will have n edges and n vertices.

We can represent our major and minor triads as triangles as in Figure 1.

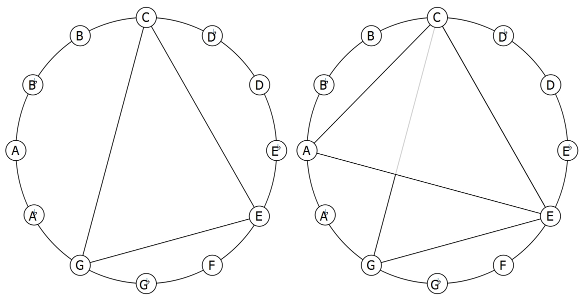

Figure 1: Tone clock representation of C (left) and a (right).

We can see that there is a relation between the two triangles. If we reflect the C triangle across the axis running through D and A♭, which bisects our major third interval between C and E, then the result is an a triangle, the parallel minor triad. There are two more relations we can find — if we reflect the triangle across the axis bisecting C and G, then we result in c, and if we reflect the triangle across the axis bisecting E and G, then we result in the relative minor, and e. We take note of these reflections over others because they result in transformations that are familiar to music theory from the common practice period.

2.3 PLR transformations

We give labels to the above canonical transformations that refer to their underlying musical relations: P is the parallel minor transformation, R, the relative minor transformation, and L for the leading-tone motion, or moving the root of the major triad down a half-step. These transformations hold for any major or minor triad and will always relate to it parallel and relative minor transformations and its leading-tone minor. We define these transformations as the following:

$$\begin{aligned} P: &\quad M \leftrightarrow \quad (m-3) \\ L: &\quad M \leftrightarrow \quad (m+4) \\ R: &\quad M \leftrightarrow \quad m \end{aligned}$$

It can be explicitly defined through modular arithmetic as:

$$\begin{aligned} P: &\quad [x, x+4, x+7] \\ &\quad \leftrightarrow [x, x+4, x+9] \mod 12 \\ L: &\quad [x, x+4, x+7] \\ &\quad \leftrightarrow [x-1, x+4, x+7] \mod 12 \\ R: &\quad [x, x+4, x+7] \\ &\quad \leftrightarrow [x, x+3, x+7] \mod 12 \end{aligned}$$

These three reflections are chosen for their voice-leading. *Parsimonious voice-leading* is when one chord is transformed into another chord by moving only one pitch and by as small an interval as possible. We see the difference between C ($[0, 4, 7]$) and $L(C) = e$ ($[4, 7, 11]$) has one voice moving one semi-tone. Any other reflection of a major triangle other than the P, L and R reflection would involve moving two or more voices, which is beyond parsimonious transformation. Here, we note that P moves one voice two semi-tones. We will later show how P is the composition of L and R transformations.

2.4 Formation of the dihedral group

THEOREM 1. *The functions P, L and R under composition as they act on the major and minor triads generate a group.*

Proof: In the proof of this theorem we follow the general scheme of Crans et al [5]. We first take note that $P^2(M) = L^2(M) = R^2(M) = M$ and $P^2(m) = L^2(m) = R^2(m) = m$, thus we have an identity function, which we identify as ϵ. Because each of our generating functions are involutions, we can see that the inverse of any composition is the "reverse" composition, i.e. $(PLR)(RLP) = (RLRP)(PRLR) = \epsilon$. Thus every function has an inverse. To show closure, we define a function to be in the group if it is a composition of P, L and R. Because each function defined has the same domain and the range is a subset of the domain, each function is well-defined. Thus the set is closed under compositions. Because compositions of functions are always associative, we have a group structure generated by P, L and R under composition. ■

By convention, we will refer to the group generated by P, L and R under composition as the PLR group. Now we will proceed to show that P is a composition of L and R and thus the PLR group can be generated by L and R.

Corollary 1. *The PLR group is generated by L and R.*

Proof: We must show that P is the composition of L and R. We observe that if we alternate R and L, we cycle through all 24 major and minor triads:

$$\mathbf{C}, a, F, d, B\flat, g, E\flat, \mathbf{c},$$
$$A\flat, f, D\flat, b\flat, G\flat, e\flat, B, a\flat,$$
$$E, d\flat, A, g\flat, D, b, G, e, C.$$

The seventh iteration of this alternating function, shown in bold above, gives us P, thus we note that $P = RLRLRLR$. Thus P is a composition of L and R and thus the group can be simply be generated by L and R. ■

Finally, we classify the PLR group as a dihedral group.

Theorem 2. *The PLR group is isomorphic to the dihedral group of order 24, D_{12}.*

Proof: First we will show that the elements of the PLR group are $\{R(LR)^n, (LR)^n | 0 \leq n \leq 11\}$. We determined that every element in the group can be generated by compositions of L and R. Because L and R are idempotent, we only consider compositions where L and R are alternating (we may replace $R \circ R$ or $L \circ L$ with the identity function). We have also discovered from the previous corollary that $(LR)^{12}$ is the identity function. Thus we will only consider compositions of alternating L and R functions that are less 24 elements long (we can replace $(LR)^{12}$ with the identity). Now, consider a function where the right-most component function is $L : (LR)^k L$. This function can be represented in the way we want, where the the right-most function is R:

$$(LR)^{12} = \epsilon$$

$$(LR)^{11} L = R$$

$$(LR)^k L = (RL)^{11-k} R$$

Thus the elements of the PLR group are $\{R(LR)^n, (LR)^n \mid 0 \leq n \leq 11\}$. We have exactly 24 elements in this group. As we recall from the definition of dihedral group, D_{12} is the group generated by two elements, s and t, such that $s^{12} = t^2 = \epsilon$ and $tst = s^{-1}$ [5]. If we let $s = LR$ and $t = L$, we will satisfy these relations: $(LR)^{12} = L^2 = \epsilon$ and $L(LR)L = RL$. Thus $PLR \approx D_{12}$. ■

2.5 Geometric representation

The common geometric representation of the PLR functions is the Oettigan-Riemann Tonnetz, also known as the Neo-Riemannian Tonnetz [5]. The Tonnetz is a two-dimensional array where the vertices represent pitch classes and triangles represent the major and minor triads. We will arrange this diagram so each triangle is equilateral, horizontal edges will represent an interval of seven semi-tones (a perfect fifth), and diagonal edges will represent thirds, those in the north-east direction (/) representing a major third and in the south-east direction (\) a minor third, as seen in Figure 2. The three pitches that form the vertices of a given triangle in the Tonnetz define a major or minor triad: CEG forms C, EGB forms e, etc.

The Tonnetz tessellates the plane perfectly, as is the nature of equilateral triangles. Because pitch-classes are cyclic (isomorphic to Z_{12}) and because we are moving by perfect fifths (equivalent to adding 7 mod 12), major thirds (adding 4 mod 12) and minor thirds (adding 3 mod 12), the ordering of major and minor triads will begin to repeat. That is to say, anywhere a C is located, it will always be surrounded by G, E, A, F, A♭ and E♭ in the very same orientation. A parallelogram formed by 12 major and 12 minor triangles will be in the same arrangement in relation to the triangles around it. The edges of the parallelogram contains the same string of pitch classes: C-E-A♭-C and C-A-G♭-E♭-C in the diagram. If we glue the edges of the parallelogram so that the edges "match up", we form a torus. Shared edges indicate two chords are related by P, L or R. Specifically, moving across the horizontal axis indicates a parallel relation, across the major third diagonal (/) indicates a relative relation, and across the minor diagonal (\) indicates an L relation.

Following Waller [9], we represent this torus as an undirected graph with vertices representing major and minor triads and an edge indicating a shared edge as shown in Figure 3. The result: four concentric cycles with interspersed bridges between the cycles. A close look at where the PLR functions are located reveals the concentric cycles are generated by alternating P and L and the bridges represent an R function. We can adjust Waller's torus so that P and R form generating cycles (creating three concentric circles), as shown in Figure 4, and most interestingly, alternating L and R which will generate a single cycle (as L and R generate the

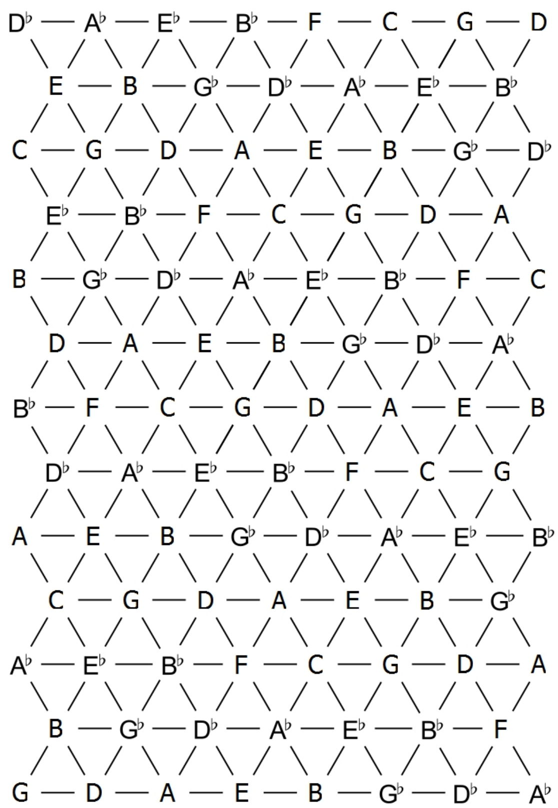

Figure 2: The Neo-Riemmanian Tonnetz depicts the pitch classes and resulting major and minor triads (triangles) so that geometric proximity is related the transformations P, L and R.

entire group).

An alternate geometric expression that may indicate the existence of a more general group structure, is the use of Coxeter kaleidoscoping [4]. A Coxeter group is generated by n elements $\{r_1, r_2, \ldots, r_n\}$. Each generator has order two ($r_i^2 = e$) and the order of each pair has the following properties: if the order of $(r_i r_j)$ is m_{ij}, then the order of $(r_j r_i)$ is also m_{ij}, $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. It is common to represent the group in terms of a Coxeter matrix, a symmetric $n \times n$ matrix with elements m_{ij}, or a Coxeter-Dynkin graph, a graph with n vertices representing the group elements and paths representing orders greater than 2. Because our generating functions (P, L and R) are idempotent, we can explore the Coxeter group structure generated by P, L and R.

Through the calculation of cycles in the PLR group,

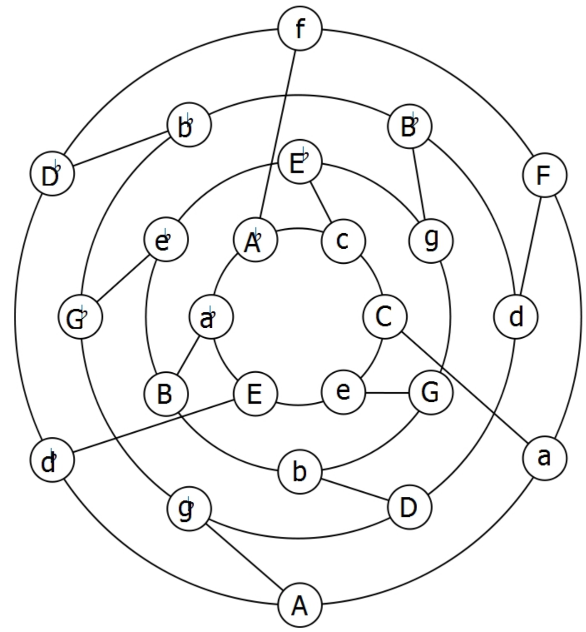

Figure 3: Waller's torus is a graphical representation of the transformations P, L and R.

we notice that PL has order 3, PR has order 4 and LR has order 12, as is seen in the Waller torus. Thus we have the following Coxeter matrix representation:

$$M = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 12 \\ 4 & 12 & 1 \end{bmatrix}.$$

The value of the path between vertices corresponds to the angle between the mirrors in a kaleidoscope representation: the angle between mirrors r_i and r_j is π/m_{ij} radians. So in our case, we have a triangle of mirrors with angles $\pi/3$, $\pi/4$ and $\pi/12$. We notice that this totals to $2\pi/3$, implying our triangle is hyperbolic. What results is this model, represented using the Poincaré disc model as shown in Figure 5.

Around each vertex in the three-functioned model, we have cycles of 24, 8 or 6 triangles. These directly correspond to alternating L and R, P and R or P and L as we noted earlier with the traditional Tonnetz. We also note that the kaleidoscoping Coxeter group reveals a common arrangement of chords into those separated by perfect fifths, often called "the circle of fifths."

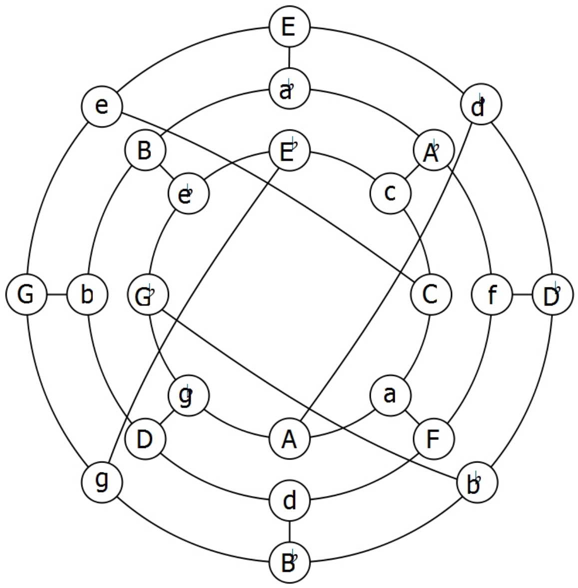

Figure 4: Three-ringed torus provides an equivalent perspective to Waller's Torus.

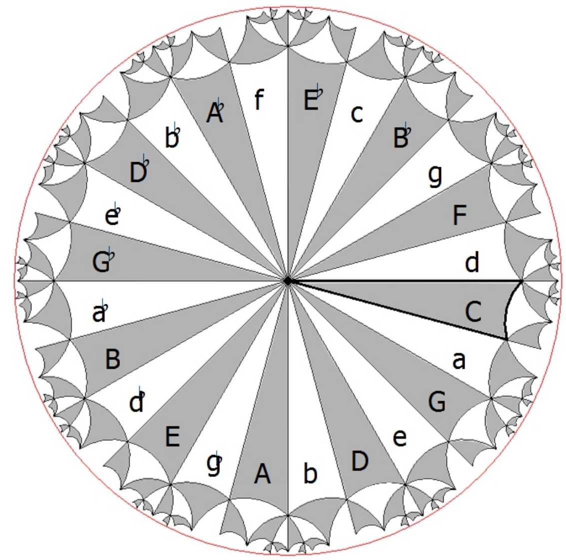

Figure 5: Hyperbolic Coxeter kaleidoscope in a Poincaré disc of P, L and R. Major-triad triangles are labeled with capital letters and minor-triad triangles are labeled with lower-case letters.

3. 3D GENERALIZATION

The two-dimensional Tonnetz, with associated graphs, hyperbolic representations, and group theoretic structure are very rich in mathematics. Our objection is that the implied musical content (for example the nearness of C major and e minor, depicted graphically in the Tonnetz and functionally by the transformation L) does not match the musical practice of tonal music, where the fundamental relationships involve cadences with chords whose the roots move by intervals of fourths and fifths. What is more, the restriction to major and minor triads is much too limited to describe tonal music, where tetrachords are ubiquitous. As early as the beginning of the 19th century, the fundamental chord types in common usage were identified by the descriptive music theorist Gottfried Weber: three triads (major, minor, diminished) and four tetrachords (dominant, minor, major, and half-diminished) [1]. In an effort to bring these considerations into better agreement, we explore generalizations of the Tonnetz to four-note structures.

Edward Gollin in 1998 examined an example of a three-dimensional expansion of the Tonnetz [6]. In two dimensions, we have two axes along which tones progress by a major third and a perfect fifth, respectively. A third direction, which can be specified as a vector difference between the two axes in the plane, progresses by a minor third.

The three-dimensional Tonnetz has three axes; the first two axes are the same as the traditional Tonnetz and the third axis along which tones progress by a minor seventh interval (10 semi-tomes). We situate the third axis so that one unit is on the perpendicular bisector of one tone length along the perfect fifth axis. This, with many other extra directions, creates a tetrahedral tessellation of Euclidean space that represent *seventh chords*, tetrachords that is composed of major and minor intervals. In particular, we create six distinct tetrahedra of the following forms:

1. * CEGB♭, with coplanar CEG and B♭ above and between C and G
2. CEGB♭ with coplanar CE and coplanar GB♭ that are skew and CE is below GB♭
3. * ACE♭G, with coplanar CE♭G and A below and between C and G
4. ACE♭G, with coplanar AC and coplanar E♭G that are skew and AC is below E♭G
5. * CE♭GB♭, with coplanar CE♭G and B♭ above and between C and G
6. CE♭GB♭, with coplanar E♭GB♭ and C below and between E♭ and B♭

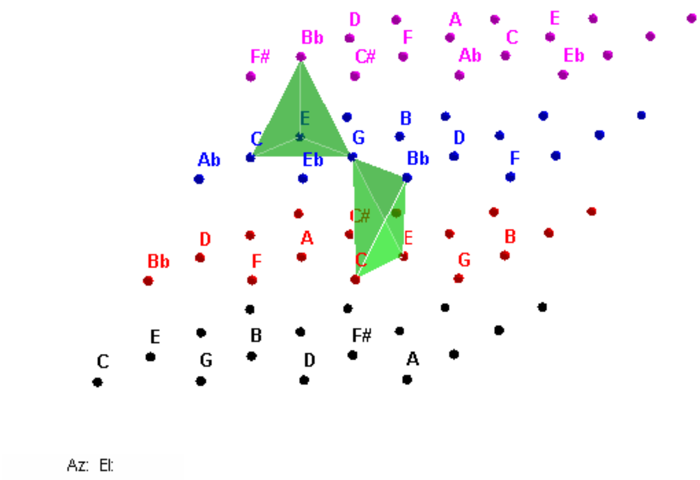

Figure 6: Three Dimensional Tonnetz showing two representations of the same dominant seventh chord, described in cases 1 and 2 in text.

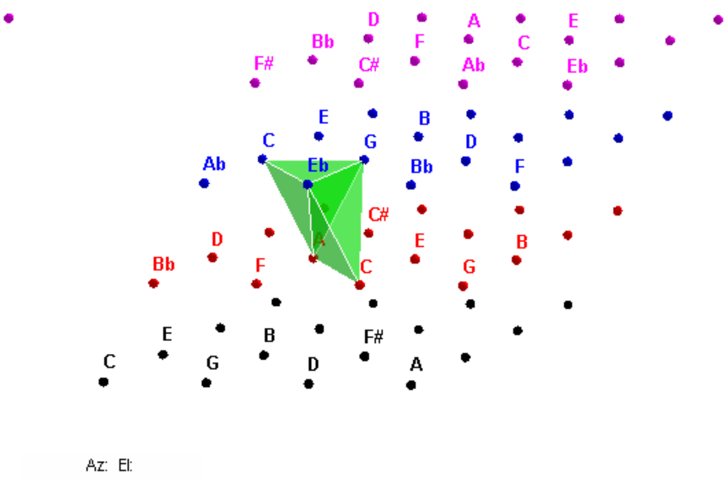

Figure 7: Three Dimensional Tonnetz showing two representations of the same half-diminished seventh chord, described in cases 3 and 4 in text.

Tetrahedra 1 and 2, which has the form of a major third, minor third and minor third in ascending order ($[0,4,7,10]$), are both known as a dominant seventh chord (C7) as shown in Figure 6. Tetrahedra 3 and 4 have the form of a minor third, minor third and major third ($[9,0,3,7]$), and are called half-diminished seventh chords (Aø7) as shown in Figure 7. The form of a minor third, major third and minor third ($[0,3,7,10]$) is a minor seventh chord (C-7), which are the forms taken by tetrahedra 5 and 6, shown in Figure 8. Each of these forms has a unique manifestation in three-dimensional space and thus each quality of chord appears in exactly two forms of tetrahedra. In this discussion, we will consider only the first of each variety (*).

Gollin explores how the dominant seven and half-diminished seventh chords are related. Using "edge-flips" and "vertex-flips", he discovers that transformations of the 24 dominant and half-diminished seventh chords create a dihedral group of order 24, exactly isomorphic to the PLR group [6]. However, this treatment omits the minor seventh chord which is very often seen in the common practice period. Thus, we will attempt to find relations among all three varieties of tetrahedra. In this exploration, we attempt to maintain the quality of parsimonious voice-leading. In our set of 36 seventh chords, there are exactly four operations that will move one voice exactly one semi-tone and will result in another element in the set:

$$\begin{array}{rrl} P1: & M7 \leftrightarrow & M-7 \\ P2: & M-7 \leftrightarrow & M\text{ø}7 \\ R1: & M7 \leftrightarrow & (M-3)-7 \\ R2: & M-7 \leftrightarrow & (M+3)\text{ø}7, \end{array}$$

or explicitly,

$$\begin{array}{ll} P1: & [x, x+4, x+7, x+10] \\ & \leftrightarrow [x, x+3, x+7, x+10] \mod 12 \\ P2: & [x, x+3, x+7, x+10] \\ & \leftrightarrow [x, x+3, x+6, x+10] \mod 12 \\ R1: & [x, x+4, x+7, x+10] \\ & \leftrightarrow [x, x+4, x+7, x+9] \mod 12 \\ R2: & [x, x+3, x+7, x+10] \\ & \leftrightarrow [x+1, x+3, x+7, x+10] \mod 12. \end{array}$$

We consider one further (pseudo-parsimonious) function, L, which transforms $M7 \leftrightarrow (M+4)\text{ø}7$. This

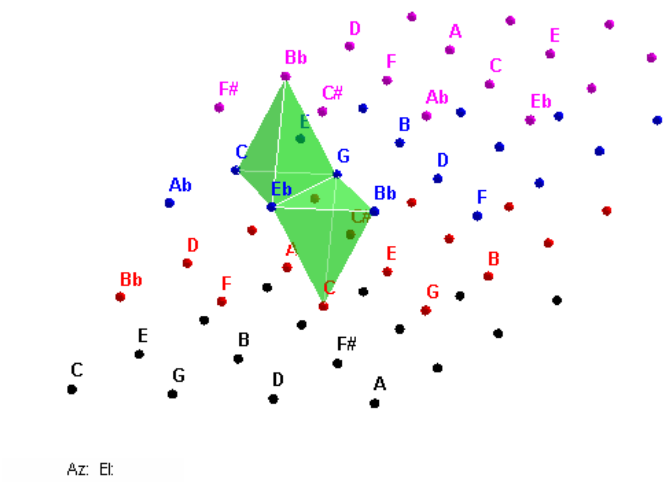

Figure 8: Three Dimensional Tonnetz showing the two representations of a minor seventh chord described in cases 5 and 6 in text.

function moves the root of the dominant seventh chord down two semi-tones and moves the seventh of the half-diminished chord up two semi-tones. With these five functions, we can create an edge-colored mapping of the 36 seventh chords in our set with each function representing a unique color, each vertex representing a chord and each edge indicating a functional relation. This mapping, we will see, fits nicely on the surface of a torus, similarly to the Waller's Tonnetz fits around a torus.

The interesting point about the two edge-colored maps we have seen, Waller's Tonnetz torus and the Seventh chord torus, is the inconsistency in the optimization of coloring. With edge-coloring we call upon Vising's theorem: a graph can be edge-colored in either the maximum degree of the graph, or one more than that [2]. The maps with the former coloring are called Class 1 graphs while the latter are Class 2. We see that in Waller's torus, each vertex is degree three and the graph is considered Class 1. However, the Seventh chord torus has inconsistent degrees, dominant and half-diminished seventh chord vertices are degree three while the minor seventh chords are degree 4. The way that the graph is colored according to our five functions, it would appear that this is a Class 2 graph. Yet, there is a recoloring, pictured below, that optimizes edge colors at four, indicating we truly have a Class 1 graph.

4. CONCLUSION

In conclusion, it would appear that there is some sort of relational structure of the tonality represented in minor, dominant and half-diminished seventh chords. The relations, however, do not parallel the two-dimensional examples of the major and minor triads. Although a group or graphical structure may not be immediately apparent through parsimonious relations, the relation between the chords is still rich for study and exploration.

4.1 Future exploration

Our work in attempting to generalize the Neo-Riemannian Tonnetz has turned out to be a deep and rich field of exploration. We leave this work with possibly more questions that we had at the beginning of the project, leading us down avenues of greater exploration.

We have implied that the parsimonious relations of the two-dimensional Tonnetz could be *coincidentally* Class 1. So the question becomes, is there a four colored edge-coloring of the edges 36 seventh chord vertices that corresponds to the musical realization of parsimonious and potentially pseudo-parsimonious voice leading? And if so, what are those functions and do they create a group with the operation of composition? Is it possible that the involutionary feature of P, L and R in the Tonnetz was merely a special characteristic of working in two dimensions or perhaps that we were only relating two flavors of chords? Could we possibly find a relation that involved compound functions that cycle through the three flavors of seventh chords we are analyzing, and cycle in the reverse when the inverse function is used? For example, what if function F transformed $M7 \to M-7 \to M\varnothing 7 \to (M-4)7$? Is this function actually relevant in the common practice period, or does it only make mathematical sense?

We may also be curious about our other geometric representations of the P, L and R. Because the major and minor triads are composed of stacked thirds, is there analogous representations using the Tonnetz with other third-stacking chords, like seventh chords? Can we find some seventh chord analogues to the tone clock, perhaps a tone sphere and using tetrahedral chords? If we can find involutionary functions relating these seventh chords, is there some kind of hyperbolic three-dimensional kaleidoscope that could contain the reflections of the tetrahedron that is reflected across a hyper-

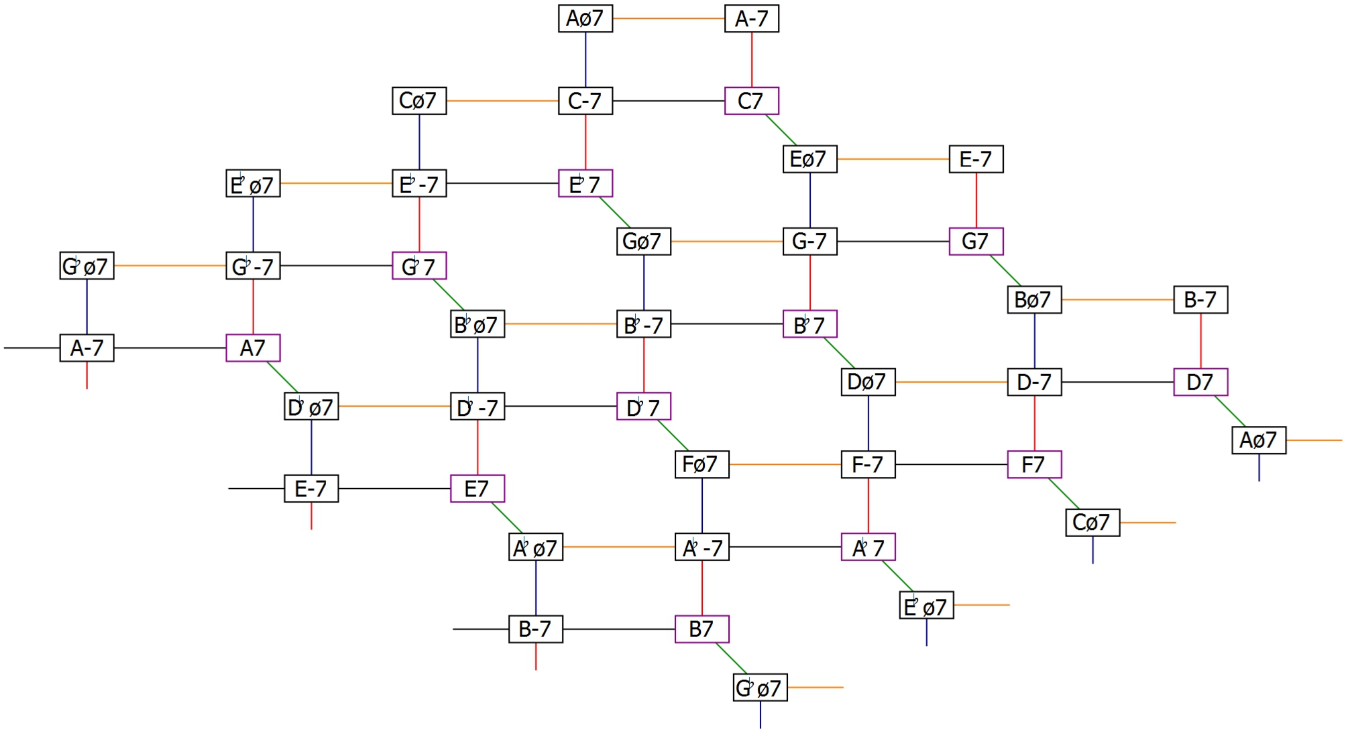

Figure 9: Graph of seventh chords edge-colored in five colors.

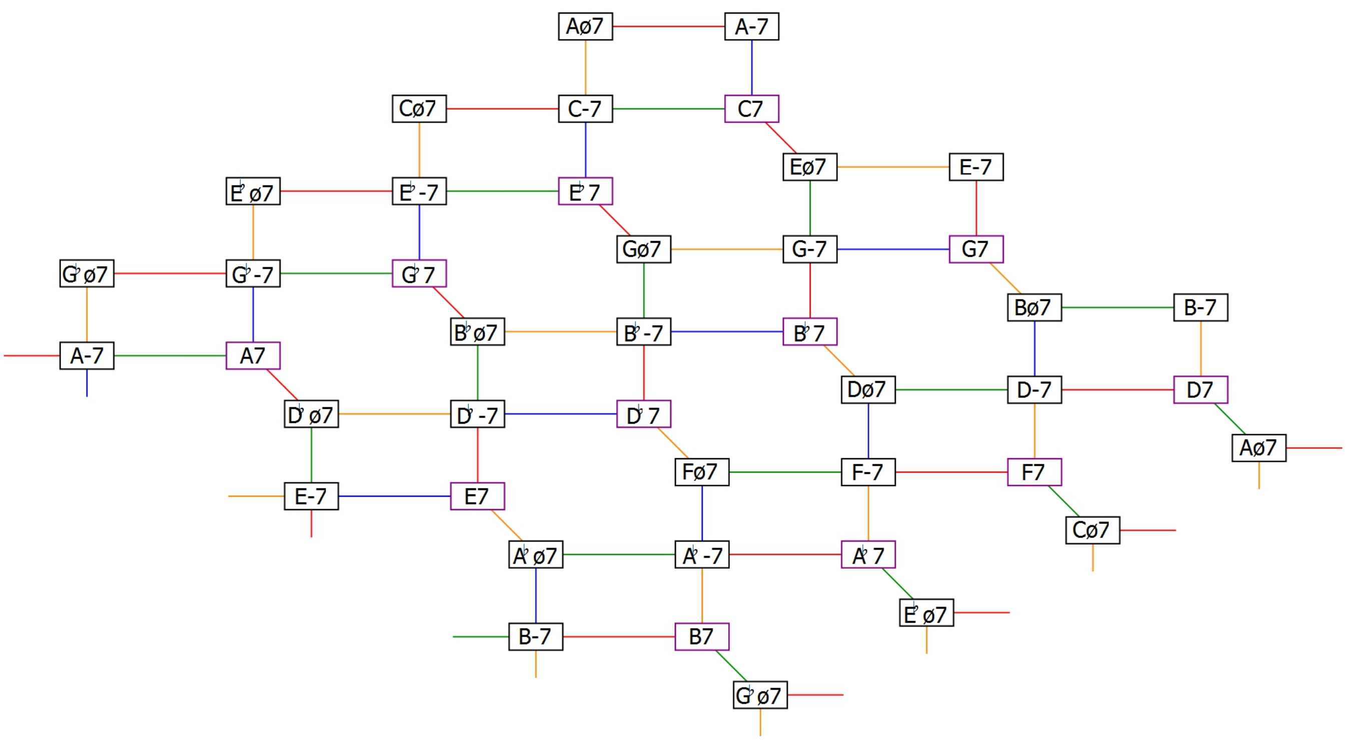

Figure 10: Graph of seventh chords edge-colored in four colors.

bolic plane?

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