Bousfield Localization and Commutative Monoids

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January 16, 2014
Big goal: understand how homotopy theory interacts with algebraic structure. We’ll use model categories and operads as our language. Notation: $\mathcal{M} =$ monoidal model category, $P =$ operad.

Subgoal: when does Bousfield localization preserve the structure of algebras over an operad?

Motivation: Hill-Hopkins-Ravenel’s resolution of the Kervaire Invariant One Problem relied on an equivariant spectrum $\Omega = D^{-1}MU^{(4)}$. When is the localization of a commutative equivariant ring spectrum commutative?
Example: Localization Can Break Commutativity

\( G = \) finite group, \( \mathcal{SP}^G = \) category of equivariant spectra.

Example presented by Mike Hill at Oberwolfach 2011:

The reduced real regular representation \( \bar{\rho} = \mathbb{R}[G]/\mathbb{R}[e] \) has representation sphere \( S^{\bar{\rho}} \). Let \( a_{\bar{\rho}} : S^0 \rightarrow S^{\bar{\rho}} \) be inclusion.

Let \( E = S^0[a_{\bar{\rho}}^{-1}] \). This is not a commutative ring spectrum! If it were, \( H < G \) would give \( N^G_H res_H E \rightarrow E \) by adjointness. But \( res_H(E) \simeq * \) so this would imply \( * \simeq E \), contradiction.

Hill-Hopkins (2013) prohibit this behavior via hypotheses on the maps being inverted. Foreshadow: What’s failing is that \( a_{\bar{\rho}} \otimes (G/H)_+ \) is not an \( a_{\bar{\rho}} \)-equivalence.
Monoidal Model Categories

1. **Pushout Product Axiom:** Given $f : A \rightarrow B$ and $g : X \rightarrow Y$ cofibrations, $f \Box g$ is a cofibration. If $f \in \mathcal{W}$ then $f \Box g \in \mathcal{W}$.

   $\begin{align*}
   A \otimes X &\longrightarrow B \otimes X \\
   &\downarrow \quad \downarrow \\
   A \otimes Y &\longrightarrow P \\
   &\downarrow \quad \downarrow \\
   B \otimes Y &
   \end{align*}$

2. **Unit Axiom:** For cofibrant $X$, $QS \otimes X \rightarrow S \otimes X \cong X$ is in $\mathcal{W}$.

3. **Resolution Axiom:** for all cofibrant $X$, $X \otimes \mathcal{W} \subset \mathcal{W}$.
Model Categories and Bousfield Localization

\[ \mathcal{M} = \text{model category}, \; \mathcal{W} = \text{weak equivalences}, \; f \notin \mathcal{W}. \]

When \( \mathcal{M} \) is left proper and either combinatorial or cellular, there is a model category \( L_f \mathcal{M} \) called the Bousfield localization of \( \mathcal{M} \) with respect to \( f \), with \( \mathcal{W}_f = \langle f \cup \mathcal{W} \rangle \supset \mathcal{W}, \; Q_f = Q, \; \mathcal{F}_f \subset \mathcal{F} \).

**Definition**

We say \( L_f \) preserves \( P \)-algebras if for all cofibrant \( E \in P-\text{alg} \), \( L_f(E) \in P-\text{alg} \) and \( E \to L_f(E) \) is a \( P-\text{alg} \) homomorphism.

More generally: given \( E \in P-\text{alg} \), we need \( \widetilde{E} \in P-\text{alg} \) with \( L_f(E) \simeq \widetilde{E} \).
Preservation of P-algebras

When P–alg inherits a model structure via $P : \mathcal{M} \rightleftarrows P–\text{alg}: U$, then fibrations and weak equivalences are created by forgetful $U$.

**Theorem (W.)**

Let $\mathcal{M}$ be a monoidal model category and let $P$ be an operad valued in $\mathcal{M}$. If $P$-algebras in $\mathcal{M}$ and in $L_f(\mathcal{M})$ inherit (semi) model structures, then $L_f$ preserves $P$-algebras.

**Proof:** $L_f(E) \simeq R_fQE$. We prove $R_fQE \simeq R_{f,P}QPE$ in $\mathcal{M}$.

![Diagram](image)

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When do $P$–algebras inherit a model structure?

**Theorem (Spitzweck, 2000)**

Suppose $P$ is a $\Sigma$-cofibrant operad and $\mathcal{M}$ is a monoidal model category. Then $P$-alg is a semi-model category which is a model category if $P$ is cofibrant and $\mathcal{M}$ satisfies the monoid axiom.

Monoid Axiom (Schwede-Shipley): Transfinite compositions of pushouts of maps in \{Trivial-Cofibrations $\otimes id_X$\} are in $\mathcal{W}$.

Genuine commutativity in $\mathcal{S}^G$ is encoded by the cofibrant operad $E_\infty^G$ with $E_\infty^G[n] = E_G\Sigma_n$ characterized by $(E_G\Sigma_n)^H = \emptyset$ if $H \cap \Sigma_n \neq \{e\}$ and $(E_G\Sigma_n)^H \simeq \ast$ otherwise. Lesser commutativity is encoded by $E_\infty^{\mathcal{F}}$.
When is $L_f(\mathcal{M})$ a monoidal model category?

Characterization of Monoidal Bousfield Localizations (W.)

$L_f(\mathcal{M})$ satisfies the Pushout Product Axiom and the Resolution Axiom iff $f \otimes K$ is an $f$-local equivalence for all cofibrant $K$.

For tractable $\mathcal{M}$ (domains of generating $I$ and $J$ are cofibrant), one need only check $K \in \{(co)domains \ of \ I \cup J\}$

Corollary

In $\mathcal{S}^G$ a Bousfield localization preserves genuine commutativity iff $f \otimes (G/H)_+$ is an $f$-local equivalence for all subgroups $H$. 

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Javier Gutiérrez and I found semi-$\mathcal{F}$-model structures on $G$–operads. The cofibrant replacements of $\text{Com}$ are Blumberg-Hill $N_\infty$ operads which we denote by $E_{\infty}^{\mathcal{F}}$. Hill’s example is maximally bad, taking $E_{\infty}^{G}$-structure down to naive $E_{\infty}$. The example generalizes to give any drop required.

Corollary

$\text{For } X \text{ above } E_{\infty}^{\mathcal{F}}$-structure, $L(X)$ has $E_{\infty}^{\mathcal{F}}$-structure iff $f \otimes (G/H)^{+}$ is an $f$-local equivalence for all subgroups $H \in \mathcal{F}$. 
Model Structure on Strict Commutative Monoids

*Commutative monoid axiom:* If $g$ is a (trivial) cofibration then $g^\Box^n / \Sigma_n$ is (trivial) cofibration. Suff. to check on generators. Stronger (Lurie, HTT): $g^\Box^n$ is a $\Sigma_n$-projective cofibration.

**Theorem (W.)**

*If a monoidal model category satisfies the monoid axiom and the commutative monoid axiom then commutative monoids inherit a model structure. Without monoid axiom it’s a semi-model structure.*

**Corollary: Preservation of Strict Commutative Monoids (W.)**

If $\text{Sym}^n(f)$ is a weak equivalence in $L_f(\mathcal{M})$ for all $n$ then $L_f(\mathcal{M})$ satisfies the commutative monoid axiom. Here $\text{Sym}(X) = S \biguplus X \biguplus X^\otimes 2 / \Sigma_2 \biguplus \cdots$
Examples: $g$ (triv) cofib $\Rightarrow g^\square_n / \Sigma_n$ (triv) cofib

Ch(k) where $\text{char}(k) = 0$. Lurie’s hypothesis holds.
sSet & Top, though they fail Lurie’s hypothesis.
Positive (Flat) model structure on symmetric spectra.
Positive orthogonal (equivariant) spectra
Positive motivic symmetric spectra - joint with M. Spitzweck.

**Corollary**

Any monoidal localization in sSet preserves commutative monoids, e.g. $L_E$ for a homology theory $E$. Truncations in sSet, Top, and Ch(k) all preserve strict commutative monoids.
Other Non-Cofibrant Operads

Harper (2010): If all symmetric sequences in $\mathcal{M}$ are projectively cofibrant then for any $P$, $P$–alg inherits a model structure.

**Theorem (W.)**

*Each row in the following table yields a semi-model structure on $P$-algebras, under a strengthened monoid axiom.*

<table>
<thead>
<tr>
<th>Hypothesis on $\mathcal{M}$</th>
<th>Class of operad</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall X \in \mathcal{M}^\Sigma_n$ projectively cofibrant, $X \otimes_{\Sigma_n} f^{\circ n}$ is a (trivial) cofibration (this follows from the pushout product axiom)</td>
<td>Cofibrant or $\Sigma$-Cofibrant</td>
</tr>
<tr>
<td>$\forall X \in \mathcal{M}^\Sigma_n$ cofibrant in $\mathcal{M}$, $X \otimes_{\Sigma_n} f^{\circ n}$ is a (trivial) cofibration</td>
<td>Levelwise cofibrant</td>
</tr>
<tr>
<td>Note: $X = *$ is the $\Sigma_n$-equivariant monoid axiom</td>
<td>Special case: $P = \text{Com}$</td>
</tr>
<tr>
<td>$\forall X \in \mathcal{M}^\Sigma_n$, $X \otimes_{\Sigma_n} f^{\circ n}$ is a (trivial) cofibration</td>
<td>Arbitrary</td>
</tr>
</tbody>
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