# The homotopy theory of ideals of ring spectra

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# Brave New Algebra

Let M be a monoidal model category of spectra. Analogy:

Classical Algebra	Brave New Algebra
Ring of Integers $\mathbb Z$	Sphere Spectrum S
Abelian groups $= \mathbb{Z}$ -modules	Spectra = $S$ -modules
(Unital) Ring	Ring spectrum
Commutative ring	$E_{\infty}$ -ring spectrum
Both: projective/injective modules,	homological dim, semisimple, etc.
Ideal $I \subset R$ s.t. $R/I$ is a ring	???

'Subobject' is the wrong idea. Better: an ideal is something you can quotient by  $I \xrightarrow{j} R \xrightarrow{\text{coker}} R/I$ . Jeff Smith (2006): an ideal is an arrow  $j: I \longrightarrow R$  with extra structure.

#### Ideals are arrows with algebraic structure

If  $(M, \otimes, 1)$  is a closed symmetric monoidal category, then the arrow category Arr(M) has two monoidal structures:

- Tensor monoidal structure:  $f \otimes g : X_0 \otimes Y_0 \longrightarrow X_1 \otimes Y_1$ , unit is  $Id_1 : 1 \longrightarrow 1$ .
- **2** Pushout product monoidal structure (unit  $\emptyset \rightarrow 1$ ):

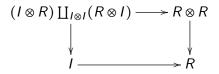
$$(X_0 \otimes Y_1) \underset{X_0 \otimes Y_0}{\coprod} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

Definition: A Smith ideal is a monoid in  $\overrightarrow{M}^{\Box} := (Arr(M), \Box)$ Note: A monoid in  $\overrightarrow{M}^{\otimes}$  is a monoid homomorphism in M.

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# Unpacking definition of Smith ideal as monoid in $\vec{\mathsf{M}}^{\square}$

A Smith ideal is a monoid R, an R-bimodule I, and a map of R-bimodules  $j: I \longrightarrow R$  such that  $\mu(1 \otimes j) = \mu(j \otimes 1) : I \otimes I \longrightarrow I$ . Reason:  $\eta : (\emptyset \longrightarrow 1) \longrightarrow j$  and unpack  $\mu : j \Box j \longrightarrow j$ :



Note: R/I is a ring spectrum and  $coker(j) : R \longrightarrow R/I$  is a ring homomorphism.

Theorem (Hovey, 2014): The cokernel functor from  $\vec{M}^{\Box}$  to  $\vec{M}^{\otimes}$  is strong symmetric monoidal  $(j \mapsto (R \longrightarrow R/I))$ , and right adjoint is the kernel. This forms a Quillen equivalence  $\vec{M}^{\Box} \leftrightarrows \vec{M}^{\otimes}$ .

### Smith's motivation: algebraic K-theory

Suppose that *R* is a ring spectrum with Smith ideals *I* and *J*. Define the Smith ideal  $I \wedge_R J$ . Let *T* be the homotopy pushout:



in the category of ring spectra. Smith: "there is a fiber sequence  $K(R/(I \wedge_R J)) \longrightarrow K(R/I) \otimes K(R/J) \longrightarrow K(T)$  of algebraic K-theory spectra." Proven by Land-Tamme, 2023; plus  $T \cong R/I \odot_R^M R/J$ , the  $\odot$ -ring from their 2019 Annals paper, for  $M = (R/I) \wedge_R (R/J)$ . Operad structure matters: in  $E_{\infty}$  context,  $A' \odot_A^M B \simeq B \odot_A^M A'$ .

# Our setup (W.-Yau)

Note: monoid morphisms are algebras over a 2-colored operad. Smith ideals are too. Generalize from *Ass* to operad *O*?

- **○** Goal: homotopy theory of ideals structured by an operad O, e.g., commutative ideals,  $A_{\infty}$ -ideals,  $E_{\infty}$ -ideals,  $E_n$ , Lie,  $L_{\infty}$ , etc. Now coker $(j) : R \longrightarrow R/I$  is O-alg morphism.
- ② Let  $L_0 \dashv Ev_0$ ,  $L_1 \dashv Ev_1$ . Given *O*, define  $\overrightarrow{O}^{\otimes} = L_0O$  (resp.  $\overrightarrow{O}^{\square} = L_1O$ ), *C*-colored operad in  $\overrightarrow{M}^{\otimes}$  (resp.  $\overrightarrow{M}^{\square}$ ).
- Solution A Smith *O*-ideal is an algebra over  $\vec{O}^{\Box}$ ; a morphism of *O*-algebras is an algebra over  $\vec{O}^{\otimes}$ .
- coker is a Quillen equiv.  $\operatorname{Alg}(\overrightarrow{O}^{\Box}; \overrightarrow{M}^{\Box}) \Leftrightarrow \operatorname{Alg}(\overrightarrow{O}^{\otimes}; \overrightarrow{M}^{\otimes})$
- There is a  $(C \coprod C)$ -colored operad  $O^s$  in M such that  $Alg(\vec{O}^{\Box}; \vec{M}^{\Box}) \cong Alg(O^s; M)$ . Use to transfer model str.

# Unpacking Smith O-ideal $j: X \longrightarrow A$ s.t. A/X is O-algebra

#### Proposition (W.-Yau)

A Smith O-ideal in M is precisely:

- an O-algebra  $(A, \lambda_1)$  in M,
- an A-bimodule  $(X, \lambda_0)$  in M, and
- an A-bimodule map  $f:(X,\lambda_0) \longrightarrow (A,\lambda_1)$

such that, for  $1 \le i < j \le n$ , the following commutes

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# What is this $O^{s}$ with $Alg(\overrightarrow{O}^{\Box}; \overrightarrow{M}^{\Box}) \cong Alg(O^{s}; M)$ ?

Given a *C*-colored operad *O*, denote by  $C^0$  (resp.  $C^1$ ) the first (resp. second) copies of  $C \coprod C$ . Given  $c \in C$ , write  $c^e \in C^e$  for the same *c* in each copy, for  $e \in \{0, 1\}$ . Define:

$$O^{s} \begin{pmatrix} c_{1}^{\epsilon_{1}} & d^{1} \\ c_{1}^{\epsilon_{1}} & \dots & c_{n}^{\epsilon_{n}} \end{pmatrix} = O\begin{pmatrix} d \\ \underline{c} \end{pmatrix}$$
$$O^{s} \begin{pmatrix} c_{1}^{\epsilon_{1}} & d^{0} \\ c_{1}^{\epsilon_{1}} & \dots & c_{n}^{\epsilon_{n}} \end{pmatrix} = \begin{cases} O\begin{pmatrix} d \\ \underline{c} \end{pmatrix} & \text{if at least one } \epsilon_{i} = 0 \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

So, an  $O^s$ -algebra is a pair (A, X) of *C*-colored objects, plus structure maps making *A* into an *O*-algebra, *X* into an *A*-bimodule, and  $f: X \longrightarrow A$  into an *A*-bimodule map.

This is similar to the two-colored operad for monoid maps.

### Main theorem

#### Theorem (W.-Yau)

If M is nice, and cofibrant Smith O-ideals are also entrywise cofibrant in  $\vec{M}^{\Box}$  then there is a Quillen equivalence

$$Smith O-Ideals \xrightarrow[ker]{coker} O-Algebra Maps$$

For  $\Sigma$ -cofibrant O, just need M stable, monoidal, cof gen. For O = Com, M needs strong commutative monoid axiom. For general O, need  $X \otimes_{\Sigma_n} (-)^{\Box n}$  and  $f \Box_{\Sigma_n} (-) : M^{\Sigma_n} \longrightarrow M$ homotopically well behaved, like preserving trivial cofibrations. Examples: symmetric spectra, Ch(k), StMod(k[G]), motivic, equivariant orthogonal spectra, enriched functors, *S*-modules, etc.

# Comparison with $\infty$ -operads

#### Theorem (W.-Yau)

If M is cof. gen.,  $M^{\flat} \subset M$ , and O is  $\Sigma_{C}$ -cofibrant (symmetric) C-colored operad.

- Denote by Alg(O; M)<sup>c</sup>[W<sub>O</sub><sup>-1</sup>] the ∞-category obtained from the semi-model category Alg(O; M), by first passing to the subcategory of cofibrant objects, and then inverting the weak equivalences between O-algebras.
- Denote by Alg(O; M[W<sup>-1</sup>]) the ∞-category obtained by first passing from M to the (symmetric) monoidal category M[W<sup>-1</sup>] and then passing to O-algebras, where O is viewed as a colored operad in M[W<sup>-1</sup>] ≃ M<sup>b</sup>[W<sup>-1</sup>].

Then  $Alg(O; M)^{c}[W_{O}^{-1}] \simeq Alg(O; M[W^{-1}])$  as  $\infty$ -categories.

# **Open Problems**

Almost every question you can ask, e.g., What is the relationship between ideals of  $\pi_*(R)$  and ideals of ring spectra? If R = S, the sphere spectrum, and  $2 \in \pi_0 S$  is the cofiber of the 'times 2' map, then (2) is an ideal of  $\pi_* S$  but the mod 2 Moore spectrum is not a ring spectrum, even up to homotopy. So what is the ring spectrum quotient of S by 2?

Let  $f: I \longrightarrow R$  be any map. What is the Smith ideal generated by f? The free functor T yields an ideal of T(R) not R. Every ring spectrum is weakly equivalent to a quotient of the sphere spectrum by some Smith ideal. Define a monoid homomorphism  $p: R \longrightarrow S$  to be a strong quotient if  $S \otimes_R QN \longrightarrow N$  is a w.e. for all fibrant N (and cof. rep. Q). Can we classify strong quotients of ring spectra? What is the connection to the 'homotopy normal maps' of Prasma?

### Work to do relating K-theory and ideals

Now is a great time to compute examples of various R/I,  $R/(I \wedge_R J)$ , and  $A' \odot_A^M B$ .

Land-Tamme is about ring spectra, but Smith ideals work in general stable model categories. Can you prove Smith's vision regarding  $E(R/(I \wedge_R J))$  for motivic spectra, equivariant spectra, chain complexes, and the stable module category? Section 6 of White-Yau lists conjectures and open problems related to Smith *O*-ideal theory in: positive flat model on symmetric spectra and equivariant orthogonal spectra, positive complete model structure, global equivariant, injective model structures, and *S*-modules.

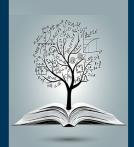
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