Talk 2: Model Categories

Motivation: Given $\mathcal{C}$ & $\mathcal{W} \subseteq \text{Mor}(\mathcal{C})$, I want to send those maps to isomorphisms.

Define $\mathcal{C}^{\mathcal{W}^{-1}}(A, B) = \frac{\mathcal{W}}{A \rightarrow B}$.

But need to identify & group with idempotents & composites.

Problem: $\mathcal{C}^{\mathcal{W}^{-1}}(A, B)$ is not a set, so can't mod out by equivalences rel. Can pass to larger Grothendieck universe, but still have little control over the maps.

It works for $\mathcal{C} = \text{Top}$ & $\mathcal{W} = \{ f \mid \pi_0 f \text{ is iso} \}$.

Get $(\mathcal{C}^{\mathcal{W}^{-1}}) = \text{HoC}$. Same obj as Top, but morphisms are homotopy classes of maps.

Reasons why it works: Top has Whitehead Thm, CW path obj, cylinders. So can define homotopies nicely & lift ho-equiv. And nice projections (fibrations).

Quillen 1967 figured out the properties to generalize.

$\mathcal{C}$ is a model category if it has $(w, c, R)$, complete, $\mathcal{W}$ 2/3, all closed under retractions, functorial factorization and lifting.

Term: "Trivial" $\mathcal{W}$, "acyclic" $\mathcal{W}$.

Covering homotopy prop, homotopy extension prop.

Any 2 of the 3 classes determine the third.

And: $\text{Ex}: \text{Top}, s\text{Set}, \text{Ch}(R), \text{S-mod}, \text{SymmCAlg}$, Motiv, Equiv, Morita's fields, retracts. Consider any morphism $f$ of schemes over $X$, $g$ on $\text{Set}$.
Constructing the Ho-Cat of a model Cat

\[ a_f : B \rightarrow X \]
A cylinder obj for \( B \) is \( B \times B / \sim \)

A path obj for \( X \) is \( \xrightarrow{\Delta} X \times X \)

A left homotopy is \( H : B \rightarrow X \) \( w / H_0 = f, H_1 = g \)
A right homotopy is \( H : B \rightarrow X' \) \( w / H_0 = f, H_1 = g \)

In gen these need not agree, but key idea is that it's ok to work up to homotopy. So replace \( B \) by cofib obj \( (\emptyset \rightarrow AB \rightarrow B) \)
replace \( X \) by fibrant \( (X \rightarrow RAX \rightarrow X) \)

Now it's an equiv. relation on \( C_{\infty} \) & \( f \sim g \Leftrightarrow f \cdot g \)

\( C(QRX, ORY) / \sim \cong HoC(XX, X'Y) \cong C(RAX, RAY) / \sim \)

Thus: \( HoC \cong C_{\infty} / / \) as categories

\( f \) is we in \( C_{\infty} \) iff \( F \cdot f \mid F \) is HoEquiv.

How To Build Them

In \( Top \), \( W \) weak hom equiv, \( F \) Serre fibrations i.e.

\[ \xrightarrow{D^n} E \]
\[ B \times I \rightarrow B \]

So somehow \( D^n \rightarrow D^n \times I \) generates \( QM^W \)

Also all \( Q \) gen by \( S^m \rightarrow D^n \times I \) 'build cells'

\( M \) is cofibrantly generated i f \( \exists \) sets of maps \( J, J, s \)

\[ F = J - \text{im} = RLP(J), W \mid F = I - \text{im} \]
\( dom(\text{I}) \) small rel I-cell,
\( dom(J) \) small rel J-cell

\( A \) is \( (J) \) small rel class of maps \( D \rightarrow F \) (A - ) conn w/ \( X \rightarrow X_0 \rightarrow X_1 \)

Where maps in \( D \) rel I-cells, transf comp of pushd maps in I
Talk 2 (cont.)

Defn of cof gen is exactly to make the small obj arg work. This is a transfinite construction of a functional factorization. You use that to build $M$ from $JFP$. You also use it to construct $Bous. Loc$ (basically).

Ex: $\text{Top}$

Ex: $\text{Set}$

Ex: $\forall i,j \in \mathbb{N}, \Delta[i] \rightarrow \Delta[j] \& \Lambda[i] \rightarrow \Delta[i]$

$J$ of $\Delta$ is Adymp. Extensions (non-red.)

Ex: $\Delta[n]$ is a closed star & $\mathbb{R}$

Omit interior of $\Delta[n]$ & $\mathbb{R}$ dim face opp.

Ex: $\mathbb{C}h(R) = \text{unbounded chain complexes}$

$I = S^{n} \rightarrow \mathbb{R}^{n}$, $J = 0 \rightarrow \mathbb{R}^{n}$

$S^{n}(M)$ is $D^{n}/M.$

Model Struct: $W =$ homology iso's

$F =$ surjections

$(Q = \text{dim. wise split, mj w/ pos. part})$

Also have new $W$, $W'$ & $Q =$ injections

Quillen Pair

Ex: Diagram Cat's w/ Rej Model Struct

$\mathbb{C} = \text{Quillen Pair}$

$F_{\mathbb{C}}$ pres $Q_{\mathbb{C}} \& ANW$

Quill. Equiv. $F_{\mathbb{C}}$ pass to equiv on $Ho$ Cat's

Derived

$HoC \xrightarrow{RF} HoD$

$LF$ is $HoC \xrightarrow{RF} HoL \xrightarrow{Q}$ $HoD$

Ex: $F = \text{id} \Rightarrow LF = Q(-)$, $RU = R(-)$
Monoidal Model Cts

Defn: Monoidal category which is model category plus coherence b/t:

1. **Product Axiom**
   - Let \( f : A \to B, g : X \to Y \)
   - \( A \otimes X \to A \otimes Y \)
   - If \( f, g \in Q \) then \( f \circ g \in Q \)
   - Further, if either for \( g \in W \) then \( f \circ g \in W \)

   (This guarantees \( \text{Hom} \) is monoidal, \([X \otimes Y] := [x] \otimes [y] \) well def)

2. **Unit Axiom**
   - Let \( S \) be unit, \( X \) cofibrant.
   - Then \( X \otimes QS \to X \otimes S = X \) is w.e.

   (This gets \( [S] \) to be unit for \( \text{Ho} \text{M} \))

- **Monoidal Functor needs** \( F \otimes FY \to F(X \otimes Y) \) \& \( FS \to S \)
- **Monoidal Quillen needs** \( FQS \to QS \)
- It's "closed" if \( \text{internal hom objects} \) \( \text{Hom}(X, Y) \in M \) plus \( \text{Hom} \) adjoint.

Dual & Equiv. Condition: \( g : W \to X, p : Y \to Z \) give

\[ \text{Hom}(X, Y) \to \text{Hom}(X, Z) \times \text{Hom}(W, Y) \]

is fibrant & triv. if either \( g \) or \( p \) is triv.

- **Ex.:** \((sSet, \times)\)
  - \((\text{Top}, \times)\) if we use compactly gen. spaces so that \( \text{Hom}(X, Y) \) gets compact-gen top

- **Ex.** \( \text{Ch}(\mathbb{R}) \) with \( (X \otimes Y)_n = \bigoplus (X_k \otimes_k Y_{n-k}) \)

\[ d(x \otimes y) = dx \otimes y + (-1)^k x \otimes dy \]

Closed Symmetric monoidal if \( \exists \text{ unital iso } \mathbb{2} (x \otimes y) \cong Y \otimes X \forall X, Y \)

- **Ex.** \( \text{Ch}(\mathbb{R}) \) has \( \mathbb{2}(x \otimes y) = (-1)^{|x|} y \otimes x \)

\[ \text{Price to move } x \text{ past } y \text{ is } \text{track of } d \text{ by } x \]

(Analogy to \( \text{Sym} \text{Spectra} \) \& \( \mathbb{E}_n, x \otimes (-) \))

In monoidal cat can define monoids & comm. monoids, if you want model cat
of Comm. mon. use Shipley's Positive Sym. Spectra (but then \( S \) not cofib). E.g. Comm