Localization and Ring Objects in Model Categories

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Localization in Algebra

Localization: systematically adjoin multiplicative inverses

Setup: $R =$ ring, $S \subset R$ multiplicatively closed

Get: $S^{-1}R = R \times S/\sim$, e.g. $(\mathbb{Z}^\times)^{-1}\mathbb{Z} = \mathbb{Q}, \langle 2 \rangle^{-1}\mathbb{Z} = \mathbb{Z}(2)$

Also get: universal ring homomorphism $R \rightarrow S^{-1}R$ taking $S$ to units, i.e. for any $f : R \rightarrow E$ taking $S$ to units

$\exists !g$ making diagram commute:

How to generalize to categories? (No mult. inverses)

Inverting $s$ is the same as inverting the map $\mu_s(r) = s \cdot r$
Localization in Categories

Setup: $C =$ category, $T =$ set of morphisms. Get: $C[T^{-1}]$ and universal $C \to C[T^{-1}]$ taking $T$ to isomorphisms.

Example: Top[\{homotopy equivalences\}^{-1}] = HoTop

Adjoining $f^{-1}$ forces us to adjoin many $g \circ f^{-1} \& f^{-1} \circ h$

$C[T^{-1}](X, Y) =$ Zigzags/$\sim$

Oops! Zigzags is not a set
Model Categories

Can’t localize an arbitrary $C$ at an arbitrary $T$

Let $C = M$ have all small (co)limits and distinguished classes of maps $W, F, Q$ satisfying some axioms.

Called: weak equivalences, fibrations (e.g. $F \to E \to B$), cofibrations (e.g. satisfying homotopy extension property)

If we set $T = W$ then $M[W^{-1}] = \text{Ho}(M)$ exists and has the desired universal property

Some model categories: Spaces, Spectra, Ch$(R)$, $G$-spectra (many model category structures)
Suppose we want to invert \( f \notin W \). Because \( \text{Ho}(\mathcal{M}) \) is nice:

\[
\begin{align*}
\mathcal{M} & \longrightarrow [L_f \mathcal{M}] \\
\downarrow & \\
\text{Ho}(\mathcal{M}) & \longrightarrow \text{Ho}(\mathcal{M})[f^{-1}] \\
\downarrow & \\
\text{obj}(L_f \mathcal{M}) = \text{obj}(\mathcal{M}) \\
L_f \mathcal{M}(X, Y) = \mathcal{M}(X, Y)
\end{align*}
\]

Under standard hypotheses on \( \mathcal{M} \), \( L_f \mathcal{M} = \) model category.

\( W_f = \langle f \cup W \rangle \supset W \), \( Q_f = Q \), \( F_f \subseteq F \)

Note: localizing a set \( T \) of maps is the same as localizing \( f = \bigsqcup_{g \in T} g \), so it’s fine to look at just \( L_f \).
A question

$L_f$ preserves many standard properties of model categories. Does it preserve monoids? Yes for $A_\infty$ and $E_\infty$. No for strict commutative (Hill, 2011). Goal: Figure out when it does.

Given associative $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ with unit $S$, a monoid $E$ has $\mu : E \otimes E \to E$, $\eta : S \to E$, commutative diagrams

\[
\begin{array}{ccc}
E \otimes E \otimes E & \rightarrow & E \otimes E \\
\downarrow & & \downarrow \\
E \otimes E & \rightarrow & E \\
\end{array}
\quad
\begin{array}{ccc}
S \otimes E & \rightarrow & E \otimes E \\
\downarrow & & \downarrow \\
E & \rightarrow & E \\
\end{array}
\quad
\begin{array}{ccc}
E \otimes S & \leftarrow & E \otimes E \\
\downarrow & & \downarrow \\
E & \rightarrow & E \\
\end{array}
\]

Morally: $a(bc) = (ab)c$ and $1 \cdot a = a = a \cdot 1$

Commutative $E$ also has twist $\tau : E \otimes E \to E \otimes E$. 
Monoidal Model Categories

1. **Pushout Product Axiom:** Given $f : A \to B$ and $g : X \to Y$ cofibrations, $f \Box g$ is a cofibration. If $f \in \mathcal{W}$ then $f \Box g \in \mathcal{W}$.

$$
\begin{array}{ccc}
A \otimes X & \longrightarrow & B \otimes X \\
\downarrow & & \downarrow \\
A \otimes Y & \longrightarrow & P \\
\downarrow & \searrow & \downarrow \\
& B \otimes Y & \\
\end{array}
$$

2. **Unit Axiom:** For cofibrant $X$, $QS \otimes X \to S \otimes X \simeq X$ is in $\mathcal{W}$

3. **Monoid Axiom:** Transfinite compositions of pushouts of maps in $\{\text{Trivial-Cofibrations} \otimes id_X\}$ are weak equivalences.

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Preservation of Strict Monoids

(1) & (2) ⇒ \text{Ho}(\mathcal{M}) is monoidal (\otimes is a Quillen bifunctor)
(3) implies the monoids \text{Mon}(\mathcal{M}) form a model category.

\( X \in \text{Ho}(\mathcal{M}) \) is a strict monoid if there is a monoid \( R \in \mathcal{M} \) commuting “on the nose” such that \( R \cong X \) in \( \text{Ho}(\mathcal{M}) \).

Localization preserves strict monoids if the composition \( \text{Ho}(\mathcal{M}) \to \text{Ho}(L_f \mathcal{M}) \to \text{Ho}(\mathcal{M}) \) takes \( X \) to a strict monoid

**Theorem**

*If \( L_f \mathcal{M} \) satisfies (1)-(3) then \( L_f \) preserves strict monoids*

\( L_f \mathcal{M} \) can fail Pushout Product Axiom: \( \mathcal{M} = \mathbb{F}_2[\Sigma_3]\)-mod and \( f : \mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \) taking 1 to \( (1, 1, 1) \)
The Unit Axiom is trivially preserved by $L_f$ because $Q_f = Q$.

**Theorem**

If $\mathcal{M}$ is a cofibrantly generated, left proper, monoidal model category with cofibrant objects flat and generating (trivial) cofibrations $I$ and $J$ having cofibrant domains, and if $f \otimes K$ is an $f$-local equivalence for all (co)domains $K$ of maps in $I \cup J$, then $L_f \mathcal{M}$ is a monoidal model category with cofibrant objects flat and domains of $I_f \cup J_f$ cofibrant.

**Theorem**

Assuming further that $\mathcal{M}$ is weakly finitely generated, that $f$ has SSet-small (co)domain, and a technical condition on $Q \otimes -$ , then $L_f \mathcal{M}$ satisfies the monoid axiom.
Preservation of Strict Commutative Monoids

**Theorem**

*If* $L_f M$ is a monoidal model category with $\text{CommMon}(L_f M)$ a model category, then $L_f$ preserves strict commutative monoids

John Harper suggested a $\Sigma_n$-equivariant monoid axiom:

Transfinite compositions of pushouts of maps $J^\Box n \otimes \Sigma_n \text{id}_X$ are in $W \forall X$

This gets $\text{CommMon}(\_)$ to be a model category, and should work for more general coloured operads

Next: $L_f$ preserves $\Sigma_n$-equivariant monoid axiom

After that: Applying results to examples, especially $G$-spectra.
References


- Carles Casacuberta, personal communication

- John Harper, personal communication


- Michael Hill, Notes from Lecture at Oberwolfach, September 19, 2011
